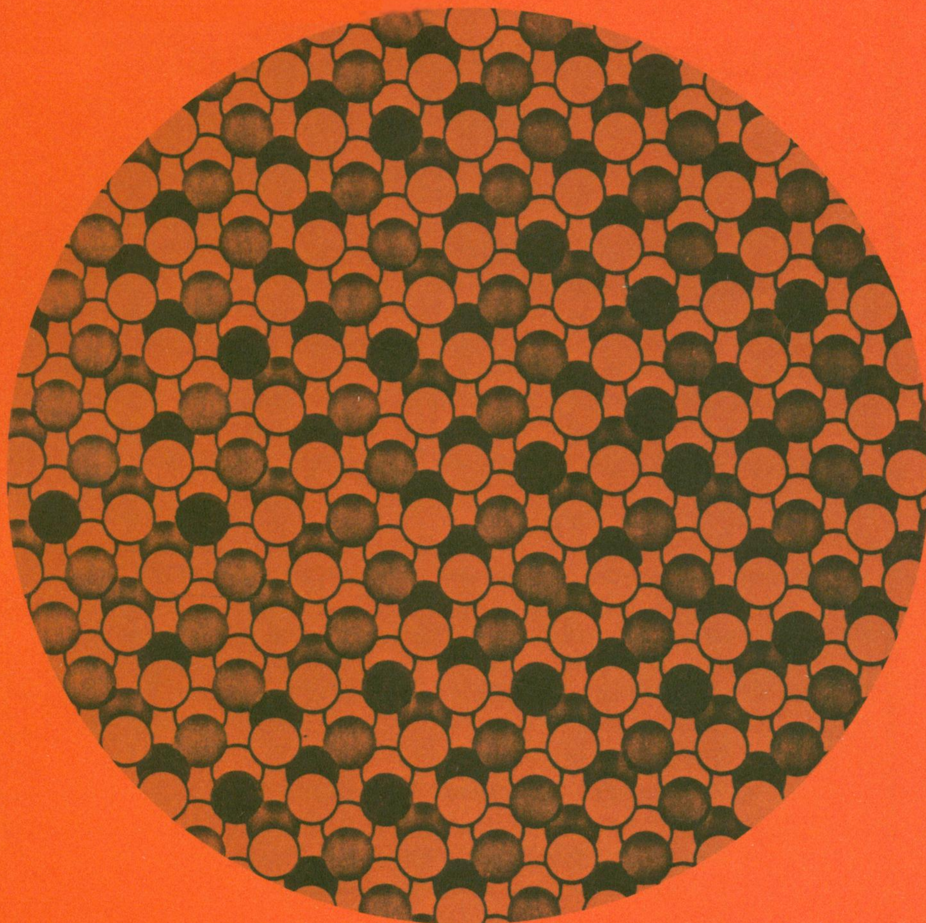
MATHEMATICS MAGAZINE



Vol. 57 No. 1
January 1984

MATRIX AND NETWORK MODELS IN ARCHAEOLOGY

REMOVING A BALL • PYTHAGOREAN TRIANGLES

MATHEMATICS MAGAZINE

THE MATHEMATICAL ASSOCIATION OF AMERICA

Volume 57, Number 1, January 1984
(ISSN— 0025-570X)

COVER: Remove a ball... See Illustrations, p. 2, and p. 27.

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS

EDITORIAL POLICY

Mathematics Magazine is a journal which aims to provide inviting, informal mathematical exposition. Manuscripts accepted for publication in the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles submitted for publication should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

New manuscripts should be sent to: Doris Schattschneider, Editor, Mathematics Magazine, Moravian College, Bethlehem, PA 18018. Manuscripts should be prepared in a style consistent with the format of *Mathematics Magazine*. They should be typewritten and double spaced on $8\frac{1}{2}$ by 11 paper. Authors should submit the original and one copy and keep one copy as protection against possible loss. Illustrations should be carefully prepared on separate sheets of paper in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Alan Shuchat ("Matrix and Network Models In Archaeology") is professor of mathematics at Wellesley College. He received his Ph.D. at the University of Michigan in 1969 under M. S. Ramanujan, taught at the University of Toledo and Mount Holyoke College before coming to Wellesley, and spent a year at the MIT Operations Research Center. His interests include operations research, functional analysis, archaeology, and playing the recorder and harpsichord.

ILLUSTRATIONS

The cover illustration is adapted from a design by Jean Larcher in *Optical and Geometrical Allover Patterns,* Dover, 1979.

All other illustrations were provided by the authors.

Authors planning to submit manuscripts should read the full statement of editorial policy which appears in this *Magazine*, Vol. 54, pp. 44–45, and is also available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

The MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, five times a year: January, March, May, September, and November.

The annual subscription price for the MATHEMATICS MAGAZINE to an individual member of the Association is $11 included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $22. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The non-member/library subscription price is $28 per year. Bulk subscriptions (5 or more copies) are available to colleges and universities for classroom distribution to undergraduate students at a 41% discount ($6.50 per copy—minimum order $32.50).

Subscription correspondence and notice of change of address should be sent to the Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036. Back issues may be purchased, when in print, from P. and H. Bliss Company, Middletown, CT 06457. Microfilmed issues may be obtained from University Microfilms International, Serials Bid Coordinator, 300 North Zeeb Road, Ann Arbor, MI 48106.

Advertising correspondence should be addressed to Ms. Elaine Pedreira, Advertising Manager, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

Second class postage paid at Washington, D.C. and additional mailing offices.

Postmaster: Send address changes to Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

PRINTED IN THE UNITED STATES OF AMERICA

Matrix and Network Models in Archaeology

The traveling salesman meets the traveling archaeologist.

ALAN SHUCHAT
Wellesley College
Wellesley, MA 02181

An important part of an archaeologist's work is to assign dates to finds, or **deposits**. For example, the deposits may be graves in a prehistoric cemetery or strata in a trench dug at the site of an ancient settlement. The deposits may contain artifacts such as pottery, jewelry, or tools. Under very fortuitous circumstances, an inscription or other unambiguous characteristic of an artifact literally labels the deposit with a date. However, such luck is not common, and so archaeologists use mathematical and statistical techniques to help solve the problem of dating deposits. This article describes the fairly recent development of the use of matrices and networks to solve an ordering problem of particular interest to archaeologists called the **seriation problem**, and illustrates these techniques with two examples from the archaeological literature. These same ideas have been successfully applied in genetics, psychology, and management. Most of the work discussed is due to D. G. Kendall, E. M. Wilkinson, and G. Laporte, but the formulations and proofs supplied here are generally simpler than the original ones. An extensive survey of the problem from an archaeologist's point of view appears in [**22**], and an overview of the application of mathematics to archaeology can be found in [**6**], [**11**], and [**3**].

The seriation problem

When archaeological deposits cannot be dated by deciphering inscriptions or applying physical or chemical techniques, but the deposits appear in stratifications, they may at least be placed in chronological order (a **seriation**). However, a site may have only partial stratification or none at all. In such cases, the archaeologist tries to order the deposits according to how their contents are related. For example, jewelry found in the deposits may show some progression of sophistication in manufacturing technique or style of decoration, and it is natural to associate this with a chronological ordering.

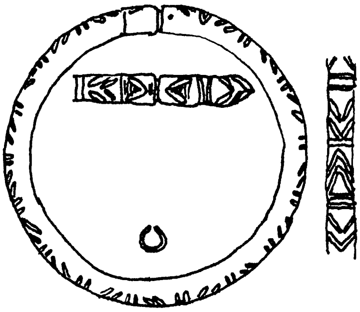
We will consider a collection of deposits containing artifacts that have been sorted into **types** by considering function, materials, shape, workmanship, style of decoration, etc. For example, a type may consist of all bronze anklets decorated with a certain arrow motif. We assume that

(**a**) each type corresponds to a time interval during which it was present, and

(**b**) each deposit corresponds to a point in time when it was formed (or at least to a time interval much shorter than the type-intervals).

Note that (**a**) rules out types that appear intermittently, such as revivals of earlier fashions.

The **seriation problem** is to order the deposits in time, based on the (as yet imprecise) notion that deposits with similar contents, as determined by the classification into types, should be closer



Bronze anklet with arrow motif, from La Tène cemetery. (Reproduced from [10].)

in time than deposits with dissimilar contents. Incidentally, if a seriation is acceptable under this standard then so is its reversal. The archaeologist needs other information to determine the orientation of a seriation in time.

Much of the work in this area during the past thirty years was stimulated by joint research of G. W. Brainerd [**4**], an archaeologist, and W. S. Robinson [**28**], a statistician. Brainerd had a collection of Mani pottery, taken from stratified deposits in three trenches dug near Mérida in the Yucatán (Mexico). There were eight deposits in all, extending over some 2,000 years. Within each trench the stratification gave the proper ordering, and Brainerd wished to merge these into a single seriation. Robinson devised a technique for ordering deposits, based on a mathematical measure of the similarity of their contents, that gave an archaeologically acceptable solution to the problem. This technique has led to more sophisticated seriation models and methods, and the Brainerd-Robinson example has often been used as a test case.

A more recently analyzed and larger example is a cemetery of the La Tène Iron Age culture, at Münsingen-Rain in Switzerland [**10**]. Here there are 214 graves arranged more or less linearly and containing, besides skeletons, objects such as jewelry and weapons. The site appears to span a period of about 500 years. The linear arrangement suggests a chronological ordering for the graves and so the cemetery has also been used as a test case for various seriation methods.

After describing matrix and network methods to solve the seriation problem, we will apply the methods to these two examples.

An incidence matrix model

The simplest representation of an archaeologist's data is in the form of an $m \times n$ matrix A of zeros and ones, where

$$m = \text{number of deposits}$$
$$n = \text{number of types}$$
$$a_{ij} = \begin{cases} 1 & \text{if deposit } i \text{ contains an artifact of type } j, \\ 0 & \text{otherwise.} \end{cases}$$

Such a matrix A is called an **incidence matrix**. Each row of A corresponds to a deposit, so the deposits are numbered according to some initial ordering. If the deposits were in their true order, we would expect the entries 1 in each column of A to be bunched roughly together. In fact, if we make the (unlikely) assumption that

(**c**) each deposit contains an example of each type extant at the time of deposit,

then (**a**) implies that for the true order, the 1's in each column are consecutive.

A **Petrie matrix** is an incidence matrix, all of whose entries 1 occur *consecutively* in each column, i.e., in each column no 0's lie between 1's [**15**]. (Flinders Petrie was a noted British archaeologist who, at the turn of the century, developed a chronology for ancient Egypt based on similar principles [**26**].) A **pre-Petrie matrix** is one whose rows can be permuted to yield a Petrie matrix. Since row permutations can be realized by multiplication on the left by a suitable permutation matrix, the seriation problem can be viewed as two matrix permutation problems **[17]**:

(I) The mathematician's seriation problem [assumes **(a)** – **(c)**]: *Given a pre-Petrie matrix A, find a permutation matrix P for which PA is a Petrie matrix. How can we recognize pre-Petrie matrices?*

(II) The archaeologist's seriation problem [assumes only **(a)** and **(b)**]: *Given an incidence matrix A, find a permutation matrix P for which PA is nearly a Petrie matrix. What should "nearly" mean?*

We can expect that several permutations will yield Petrie or nearly Petrie matrices. In practice, the archaeologist will have to use additional information to choose among them.

In the Brainerd-Robinson study, the trenches were denoted by Roman numerals and the deposits within a trench were labeled alphabetically according to the stratigraphy, with A being the uppermost and thus most recent deposit. The initial ordering of the eight deposits was IIIC, IIIB, IIIA, IB, IA, IIC, IIB, IIA. The pieces of pottery found in the deposits were sorted into eight types. For the incidence matrix, it is convenient to add a row of zeros corresponding to $i = 0$. For this example, the incidence matrix A is the following 9×8 matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{1}$$

In the Münsingen-Rain study, F. R. Hodson **[10]** identified 73 types of jewelry and 65 "diagnostic" graves that contained at least two jewelry types. J. E. Doran **[5]** reduced the data to $m = 63$ deposits and $n = 69$ types by eliminating uncertain classifications. By adding a row of zeros, a 64×69 incidence matrix is obtained. This is the largest of 31 seriation problems analyzed by Laporte **[18]**. In most of these problems, $m \leqslant 15$ and $n \leqslant 25$.

Network models

The matrix problems just described can be transformed into network problems in which one looks for minimal length circuits. Let A be an $(m+1) \times n$ incidence matrix with an initial row $(i = 0)$ of zeros and $S = (s_{ij})$ the $(m+1) \times (m+1)$ symmetric matrix defined by

$$S = AA^t. \tag{2}$$

Since s_{ij} is the number of types that deposits i and j have in common, we call S a **similarity** (or **correlation**) **matrix**. For the Brainerd-Robinson problem, whose matrix A is given by (1), S is the 9×9 matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 2 & 5 & 5 & 2 & 1 & 2 \\ 0 & 5 & 8 & 4 & 6 & 7 & 4 & 3 & 5 \\ 0 & 2 & 4 & 4 & 3 & 4 & 3 & 1 & 4 \\ 0 & 5 & 6 & 3 & 6 & 6 & 3 & 2 & 3 \\ 0 & 5 & 7 & 4 & 6 & 7 & 4 & 2 & 4 \\ 0 & 2 & 4 & 3 & 3 & 4 & 4 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 & 2 & 2 & 3 & 2 \\ 0 & 2 & 5 & 4 & 3 & 4 & 3 & 2 & 5 \end{bmatrix}. \tag{3}$$

It is easy to see from definition (2) that for all $i, j, 0 \leqslant s_{ij} \leqslant n$, so

$$d_S(i, j) = n - s_{ij} \tag{4}$$

is a measure of dissimilarity (although d_S is not a metric on the rows of A since even if rows i and j are identical, $d_S(i, j)$ need not be zero).

Now form a weighted, undirected graph (network) whose vertices correspond to the $m+1$ rows of A (the deposits) and connect each pair of vertices i, j by an edge whose **length** (weight) is given by $d_S(i, j)$ [**27**, Chap. 4]. Row 0 of A corresponds to a "dummy" vertex in the network whose distance from each other vertex is n. For example, the network N corresponding to the matrix S in (3) is shown in FIGURE 1.

There is a one-to-one correspondence of the permutations of the original rows of A with those of A that fix row 0, and thus with the **Hamiltonian circuits** in N, i.e., the closed paths in N that visit each vertex once. Let P denote both such a permutation and its permutation matrix, i.e., $P(j)=i$ means that the ith row of the permuted incidence matrix PA is the jth row of A. Then $(0=P^{-1}(0), P^{-1}(1),\cdots,P^{-1}(m),0)$ is the corresponding Hamiltonian circuit, which we also denote by P. For example, when A is the matrix in (1) then the permutation $P=(2\ 3)(5\ 6\ 7)$ corresponds to the circuit $P=(0,1,3,2,4,7,5,6,0)$ in FIGURE 1. If $B=PA$, then the similarity matrix $T=BB^t$ of B is just PSP^t, the result of using P to permute both the rows and columns of S. This observation shows that the networks given by T and S are isomorphic, i.e., they are the same except for the labels on their vertices. For a Hamiltonian circuit P in N, we define the length $L(P)$ as the sum of the lengths of the edges in P, i.e.,

$$L(P)=\sum_{i=0}^{m} d_T(i,i+1), \tag{5}$$

where $T=PSP^t$ is the similarity matrix associated with PA (for $i=m$, set $i+1=0$).

Let $\operatorname{tr} S$ denote the **trace** of S, i.e., the sum of entries on the main diagonal of S. From the definition of S in (2) and the fact that A is an incidence matrix, it follows that

$$s_{ii}=\sum_j a_{ij}a^t_{ji}=\sum_j a^2_{ij}=\sum_j a_{ij},$$

which is the ith row sum of A. Thus $\operatorname{tr} S$ is just the total number of 1's in A and so is invariant under permutations, i.e., $\operatorname{tr} T=\operatorname{tr} S$.

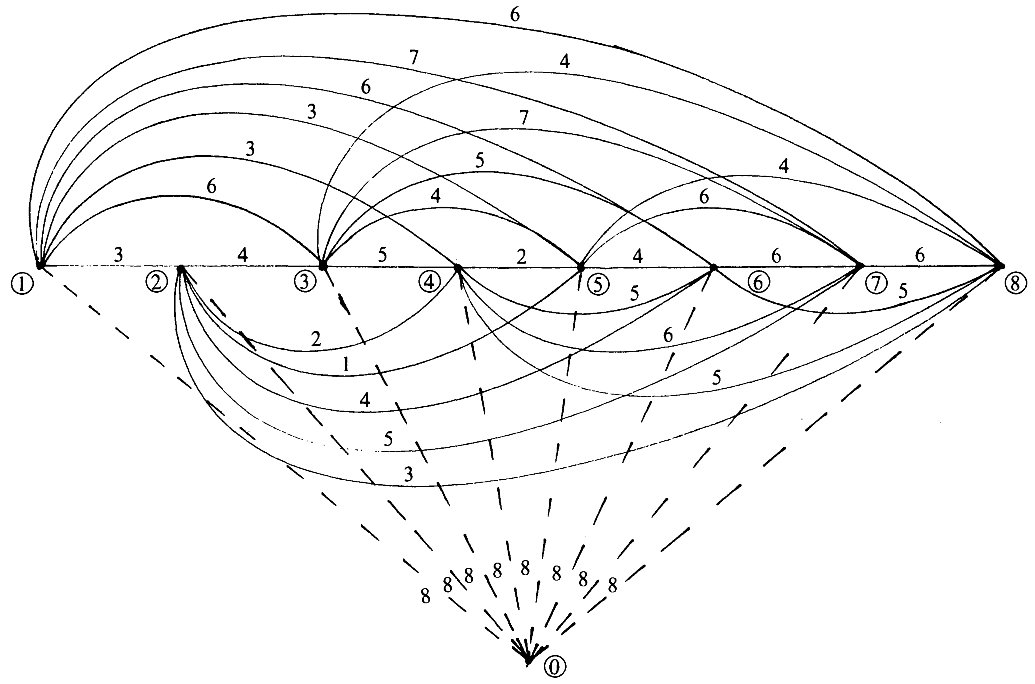


FIGURE 1. **The network for (3) with arcs (i, i) omitted.**

To find the Hamiltonian circuit of shortest length in such a network is called the **(symmetric) traveling salesman problem (TSP)**, an important problem in graph theory and operations research. Its name refers to a person who wants to plan a route that passes through each town in a sales territory exactly once and then returns to the home office. The problem is symmetric in that there are no one-way roads, i.e., the graph is undirected. The following theorem shows how the seriation problem is related to the **TSP**.

THEOREM 1. *If A is an $(m+1)\times n$ incidence matrix with an initial row of 0's, S its similarity matrix, and N the associated network described above, then each Hamiltonian circuit P in N has length*

$$L(P) \geqslant (m+2)n - \operatorname{tr} S.$$

A is a pre-Petrie matrix if and only if this length is attained by some P, in which case PA is a Petrie matrix.

Proof. Let $B = PA$ and $T = BB^t$. Then

$$L(P) = \sum_{i=0}^{m} d_T(i, i+1) = (m+1)n - \sum_{j=1}^{n}\left[\sum_{i=0}^{m} b_{ij}b_{i+1,j}\right]. \tag{6}$$

Since the bracketed sum, which we call C_j, is the number of pairs of consecutive 1's in the jth column of B and

$$C_j \leqslant \left(\sum_{i=0}^{m} b_{ij}\right) - 1, \tag{7}$$

we obtain

$$L(P) \geqslant (m+2)n - \operatorname{tr} S.$$

A circuit can attain this bound if and only if (7) is an equality for each j, i.e., if and only if the 1's in each column of B are consecutive.

If we apply Theorem 1 to the Brainerd-Robinson example, with matrix S given by (3), we see that $(m+2)n - \operatorname{tr} S = 38$ is a lower bound for $L(P)$; this will provide a test for determining whether A in (1) is a pre-Petrie matrix. The initial ordering corresponds to $P = I$, the identity matrix, and a circuit of length $L(I) = 46$.

Theorem 1 was first proved by Wilkinson [**30**] using the l^1 (or "taxicab") metric on the rows A_i of A to assign lengths to the edges of N and later by Laporte [**20**] who used the function d_S. Since the l^1 metric is defined by

$$\delta(A_i, A_j) = \sum_k |a_{ik} - a_{jk}|,$$

it gives the number of types present in one deposit but not the other. So

$$\delta(A_i, A_j) = s_{ii} + s_{jj} - 2s_{ij},$$

which makes it easy to translate Theorem 1 into l^1 terms: the minimum circuit length becomes $2n$. We use d_S rather than δ because it simplifies the discussion below. Theorem 1 reduces the mathematician's seriation problem **(I)** to solving a **TSP**. However, if A is not a pre-Petrie matrix, the permutation corresponding to a shortest circuit need not give a seriation acceptable to the archaeologist, as we now see.

Let $B = PA$. Then $L(P) - [(m+2)n - \operatorname{tr} S]$ is one measure of the deviation of B from the Petrie form. By (6) and (7), we have

$$L(P) - [(m+2)n - \operatorname{tr} S] = \sum_{j=1}^{n}\left[\sum_{i=0}^{m} b_{ij} - C_j\right] - n.$$

Each column of B consists of alternating groups of 1's and zeros. In each group of 1's,

$$(\text{number of 1's}) - (\text{number of pairs of consecutive 1's}) = 1$$

so in the jth column,

$$\sum_{i=0}^{m} b_{ij} - C_j = \text{number of groups of 1's}$$

$$= \text{number of groups of embedded zeros} + 1,$$

i.e., zeros embedded between 1's.

Summing over j, we obtain

$$L(P) - [(m+2)n - \operatorname{tr} S] = \text{number of groups of embedded zeros in } B, \tag{8}$$

so finding a circuit of shortest length amounts to minimizing this value. Formula (8) counts the number of violations of the Petrie form without regard to their "duration." Especially for large matrices, it seems more reasonable to minimize the total number of embedded zeros instead. This is what we do next.

For a given matrix $B = PA$, let

$$r_j = (\text{row index of the last 1 in column } j) - (\text{row index of the first 1 in column } j) \tag{9}$$

and let

$$R(P) = \sum_{j=1}^{n} r_j \tag{10}$$

be the **score** of P. It is clear that minimizing $R(P)$ means minimizing the number of embedded zeros in B.

THEOREM 2. *Each Hamiltonian circuit P in N has score*

$$R(P) \geqslant \operatorname{tr} S - n.$$

A is a pre-Petrie matrix if and only if this score is attained by some P, in which case PA is a Petrie matrix.

Proof. For column j of $B = PA$, (9) implies that

$$r_j = \text{number of 1's} + \text{number of embedded zeros} - 1.$$

So for the entire matrix B, (10) implies

$$R(P) = \operatorname{tr} S - n + \text{total number of embedded zeros in } B, \tag{11}$$

which gives the desired result.

For the Brainerd-Robinson example, Theorem 2 tells us that $\operatorname{tr} S - n = 34$ is a lower bound for $R(P)$. The initial ordering has score $R(I) = 46$.

The score (10) and length (5) are related in the following way. For a Hamiltonian circuit P and $B = PA$, let

$$C(P) = \sum_{j=1}^{n} C_j \tag{12}$$

be the total number of pairs of consecutive 1's in B. By (6), $L(P) = (m+1)n - C(P)$, so finding the shortest Hamiltonian circuit is equivalent to maximizing $C(P)$. We can combine Theorems 1 and 2 to obtain the following.

THEOREM 3. *Each Hamiltonian circuit P in N satisfies*

$$C(P) \leqslant \operatorname{tr} S - n \leqslant R(P).$$

A is a pre-Petrie matrix if and only if some P satisfies one of these inequalities exactly, in which case it satisfies the other exactly and PA is a Petrie matrix.

Incidentally, $C(P)$ can also be computed directly from the permuted similarity matrix $T = BB^t$:

$$C(P) = \sum_{i=0}^{m-1} t_{i,i+1},$$

the sum along the super (or sub)-diagonal of T. For the Brainerd-Robinson initial ordering, $C(I) = 26$.

We call the problem of finding a Hamiltonian circuit P that minimizes $R(P)$ the **traveling archaeologist problem** (**TAP**). Doran [**5**] was apparently the first to propose its solution as a method for dealing with non-pre-Petrie incidence matrices. Laporte [**19**] presents computational evidence that for sufficiently large matrices, minimizing $R(P)$ gives better results than minimizing $L(P)$ (see, however, [**31**, pp. 21–22] for an objection on theoretical grounds).

Theorem 2 reduces both problems (**I**) and (**II**) to solving a **TAP**. The extent to which the permuted matrix PA for a circuit P of minimal score $R(P)$ deviates from the mathematically ideal Petrie form is measured by $R(P) - [\mathrm{tr}\, S - n]$, the total number of embedded zeros in PA.

We discuss later how the **TAP** and **TSP** can be solved, but now only mention that they are computationally difficult and for large networks an approximate rather than an exact solution may be all that is feasible. Results for the 63-deposit Münsingen-Rain problem support the claim that the **TAP** model is superior to the **TSP** model for larger problems. We consider three seriations:

(H) Hodson's seriation by traditional archaeological methods [**10**],
(TAP) the best known seriation by the **TAP** criterion [**5**], and
(TSP) the optimal solution by the **TSP** criterion [**18**].

The orderings of (H) and (TAP) turn out to be fairly similar, while that of (TSP) is quite different, roughly reversing the order of the first 20 graves and placing most of the last 25 somewhere in the middle [**19**]. Incidentally, the (TAP) seriation was found by improving by hand an approximate computer-generated solution. TABLE 1 shows that the score $R(P)$ does a better job of discriminating among these orderings than $L(P)$ or $C(P)$.

Seriation	$R(P)$	$L(P)$	$C(P)$
H	456	4293	123
TAP	414	4288	128
TSP	829	4265	151

TABLE 1. **Seriations for the Münsingen-Rain problem using the incidence matrix model.**

For the Brainerd-Robinson problem, the minimum score $R(P)$ is 38 (see [**18**]), corresponding to the seriation

$$\mathrm{IIIC, IB, IA, IIIB, IIA, IIIA, IIC, IIB.} \tag{13}$$

(Compare this with the initial ordering, p. 5.) Since the lower bound of 34 for $R(P)$ is not attained, the incidence matrix (1) is not a pre-Petrie matrix. Seriation (13) also gives the minimum length $L(P) = 40$. However, we must reject (13) as a realistic solution since it places the uppermost deposit in the second trench, IIA, earlier than the two below it. There are other seriations with scores close to 40 for the archaeologist to consider. For instance,

IIIC	IB	IA	IIIB	IIIA	IIC	IIB	IIA	(14)
IIC	IIB	IB	IIIC	IA	IIIB	IIA	IIIA	(15)
IIC	IIB	IB	IIIC	IA	IIIB	IIIA	IIA	(16)

all give the proper order for each trench and are good candidates for the true ordering. TABLE 2 shows that $R(P)$, $L(P)$, and $C(P)$ fail to discriminate well among these candidates.

Seriation	$R(P)$	$L(P)$	$C(P)$
(13)	38	40	32
(14)	41	43	29
(15)	42	42	30
(16)	43	43	29

TABLE 2. **Seriations for the Brainerd-Robinson problem using the incidence matrix model.**

A stochastic matrix model

The results obtained for the Brainerd-Robinson problem, using the techniques discussed, are fairly inconclusive. This prompts us to ask if we discarded too much of the available information in representing the archaeological data by an incidence matrix. An incidence matrix treats all types found in a deposit equally, which may not always be appropriate. For example, if the deposits are not completely isolated from one another, then some artifacts may have been assigned to the wrong deposit. This may have happened with the Brainerd-Robinson strata but probably not with the Münsingen-Rain graves. Also, the effect of an error is likely to be greater in a small problem than in a large one. In such cases it would be better to give more weight to types that are heavily represented and less to ones that are not.

We define an $m \times n$ matrix $F = (f_{ij})$ where

$$f_{ij} = \text{fraction of deposit } i \text{ consisting of type } j.$$

The rows F_i of F consist of nonnegative entries whose sum is 1 and so we may think of the F_i as probability vectors: f_{ij} is the probability that a randomly selected artifact at time i is of type j. Such a matrix F is called a **(row-) stochastic matrix**. It is often reasonable to assume that in a culture, types come into fashion, increase in popularity to some peak, then fade and disappear. Thus if the deposits were in their true order, we would expect the entries in each column of F to be non-decreasing up to some maximum and non-increasing afterwards. We call such column vectors **unimodal** and say F is a **unimodal matrix** if its columns are unimodal vectors. We call F **pre-unimodal** if PF is unimodal for some permutation matrix P. As before, add a row of zeros $(i = 0)$ to F.

For example, here is the stochastic matrix F in the Brainerd-Robinson problem, for the original order in (1):

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .066 & .033 & .055 & .275 & .571 & 0 \\ .543 & .035 & .140 & .018 & .053 & .070 & .123 & .018 \\ .296 & 0 & .141 & 0 & 0 & .070 & 0 & .493 \\ .003 & 0 & .002 & .002 & .005 & .014 & .974 & 0 \\ .113 & 0 & .038 & .013 & .033 & .249 & .526 & .028 \\ .002 & 0 & .002 & 0 & 0 & 0 & .993 & .003 \\ .014 & .009 & 0 & 0 & 0 & 0 & .977 & 0 \\ .240 & .668 & .013 & 0 & 0 & .040 & 0 & .039 \end{bmatrix}. \qquad (17)$$

The seriation problem for stochastic matrices can be expressed as follows.

(I′) Given a pre-unimodal stochastic matrix F, find a permutation matrix P for which PF is unimodal. How can we recognize pre-unimodal matrices?

(II′) Given a stochastic matrix F, find a permutation matrix P for which PF is nearly unimodal. What should "nearly" mean?

In the reformulation of our seriation problem as **(I′)**, **(II′)**, the stochastic matrix F replaces the incidence matrix A. We follow **[16]** and define a symmetric matrix U which serves a comparable function to that of the matrix $S = AA^t$ in our earlier approach. The matrix $U = (u_{ij})$ has entries

$$u_{ij} = \sum_k \min(f_{ik}, f_{jk}).$$

Then $0 \leqslant u_{ij} \leqslant 1$; deposits i and j are disjoint if and only if $u_{ij} = 0$ and identical if and only if $u_{ij} = 1$. We define

$$d_U(i, j) = 1 - u_{ij}$$

as a measure of dissimilarity and, unlike the case with incidence matrices, d_U is a metric on the set of deposits. In fact,

$$d_U(i, j) = \frac{1}{2}\delta(F_i, F_j) = \frac{1}{2}\sum_k |f_{ik} - f_{jk}|, \tag{18}$$

so we will use δ instead of d_U.

Now form a network N' whose vertices correspond to the rows of F and connect vertex i to vertex j by an edge whose length is $\delta(F_i, F_j)$. Row 0 corresponds to a "dummy" vertex at a unit distance from the others. As before, permutations of the original rows of F correspond to permutations that fix row 0 and thus to Hamiltonian circuits in N'. Also as before, the length of a Hamiltonian circuit P in N' is defined as the sum of the lengths of the edges of P, i.e.,

$$L'(P) = \sum_{i=0}^{m} \delta(G_i, G_{i+1}), \text{ where } G = PF.$$

THEOREM 4. *Every Hamiltonian circuit P in N' has length*

$$L'(P) \geqslant 2\sum_j \max_i f_{ij}.$$

F is pre-unimodal if and only if this length is attained by some circuit P, in which case PF is unimodal.

Proof. Let $G = PF$ and $V = PUP^t$. Then

$$\begin{aligned} L'(P) &= \sum_{i=0}^{m} \delta(G_i, G_{i+1}) \ (\text{set } m+1 = 0) \\ &= \sum_{i=0}^{m} \sum_{j=1}^{n} |g_{ij} - g_{i+1,j}|. \end{aligned} \tag{19}$$

Fix j and choose $k \geqslant 1$ so that $g_{kj} = \max_i g_{ij} = \max_i f_{ij}$. If column j is unimodal, then

$$\begin{aligned} \sum_{i=0}^{m} |g_{ij} - g_{i+1,j}| &= \sum_{i=0}^{k-1} (g_{i+1,j} - g_{ij}) + \sum_{i=k}^{m} (g_{ij} - g_{i+1,j}) \\ &= 2g_{kj}, \end{aligned} \tag{20}$$

since $g_{0j} = 0$. If column j is not unimodal, then some terms on the right in (20) are negative and the left side must exceed $2g_{kj}$. The result now follows.

Theorem 4, first proved by Wilkinson [**30**], thus reduces the mathematician's problem (**I′**) for seriation with stochastic matrices to solving a **TSP**. Unlike the solution with incidence matrices, the **TSP** model also gives a reasonable solution to the archaeologist's problem (**II′**). Indeed, by (19) and (20),

$$L'(P) - 2\sum_j \max_i f_{ij} = 2\sum_j \sum_* |g_{i+1,j} - g_{ij}|,$$

where * indicates those values of i (depending on j) for which the unimodal form is violated, i.e.,

$$g_{i+1,j} < g_{ij} \qquad \text{for } i \leqslant k-1,$$

$$g_{i+1,j} > g_{ij} \qquad \text{for } i \geqslant k.$$

Thus $L'(P) - 2\Sigma_j \max_i f_{ij}$ is a reasonable measure of the deviation of $G = PF$ from the desired unimodal form and finding the Hamiltonian circuit of shortest length will solve both (**I**′) and (**II**′).

For the Brainerd-Robinson problem with F as given in (17), the solution to the corresponding **TSP** problem is seriation (16) (see [**13**]). Robinson's solution was also (16) (see [**28**]). In fact, TABLE 3 shows that with the stochastic matrix model, (15) and (16) are substantially better orderings than (13) and (14).

Seriation	$L'(P)$
(13)	9.56
(14)	9.83
(15)	7.01
(16)	6.65

TABLE 3. **Seriations for the Brainerd-Robinson problem using the stochastic matrix model.**

Solving the TSP and TAP

The **TSP** is an example of an **integer programming** problem, i.e., it is a linear programming problem (a linear function must be maximized or minimized subject to linear equality or inequality constraints) in which the variables must be integers. For example, the **TSP** for the stochastic matrix model can be expressed in the following form, where the variables q_{ij}, $i<j$, must be 0 or 1, and the subscripts denote vertices:

$$\begin{aligned} &\text{minimize} && \sum_{i,j} \delta(F_i, F_j)\, q_{ij} \\ &\text{subject to} && \sum_{i<k} q_{ik} + \sum_{j>k} q_{kj} = 2, \quad \text{all } k \\ & && \sum_{\substack{i \in X \\ j \in \bar{X}}} q_{ij} \geqslant 2, \end{aligned}$$

for all complementary sets X, $\bar{X}$ of vertices (i.e., $X \cup \bar{X}$ is the complete set of vertices of N') [**23**]. Values of q_{ij} that satisfy the constraints determine a Hamiltonian circuit: $q_{ij} = 1$ if and only if the circuit includes the edge between vertex i and vertex j. Since the seriation corresponding to permutation P is $P^{-1}(0)$, $P^{-1}(1), \dots, P^{-1}(m)$, we can obtain the permutation matrix $P = (p_{ij})$ corresponding to the values of q_{ij} by setting $P^{-1}(0) = 0$ and

$$p_{ij} = 1 \Leftrightarrow j = P^{-1}(i) \Leftrightarrow q_{P^{-1}(i-1),\,j} = 1,$$

where for $i = 0$ we take $i - 1 = m$; $p_{ij} = 0$ otherwise.

The **TAP** can be expressed in a somewhat similar fashion [**18**] as a **mixed-integer programming problem**, i.e., a linear programming problem where only some of the variables must be integers.

Both the **TSP** and **TAP** are examples of **NP-complete** problems [**8**], a class of problems for which (a) no efficient algorithm is known (i.e., an algorithm whose running time is a polynomial function of the size of the problem, in this case the number of vertices) and (b) knowing whether such a problem can be solved efficiently determines whether many other such problems can be solved efficiently.

Optimization problems are solved by either **exact** or **heuristic** methods. For the **TSP** and **TAP**, exact algorithms minimize the length or score over all circuits, preferably without examining all circuits individually, while heuristic algorithms minimize it only over some "reasonable" subset of circuits. There is a rich literature on heuristics for the **TSP**. A good account is given in reference [**25**]; the reader can find an account of actual attempts on large **TSP** in [**21**].

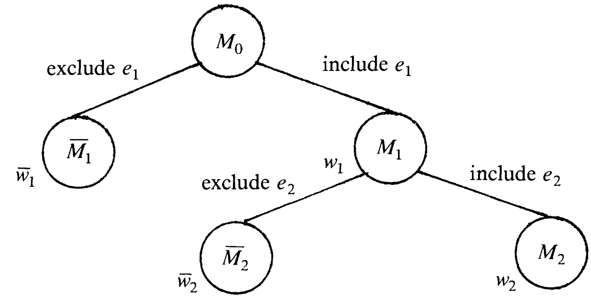


FIGURE 2. **Branch-and-bound tree.**

Problems as large as the 64-vertex Münsingen-Rain network, the largest archaeological example studied in [**18**], are usually solved by heuristics, although recent articles report exact solutions for symmetric **TSP**'s with over 100 vertices [**9**], [**24**]. The **TAP** solution for the Münsingen-Rain problem described above was a heuristic one. Apparently, no one has attempted to seriate the Münsingen-Rain data using a **TSP** algorithm and a stochastic matrix model. Shen Lin's algorithm [**21**] may be useful here.

We will consider here just one popular type of exact algorithm for integer programming called **branch and bound**, and see that it can be used for the **TAP** as well as the **TSP**. We begin by choosing one edge e_1 in our network (N or N') and dividing the set M_0 of all Hamiltonian circuits into two parts: the set M_1 of circuits containing e_1 and its complement $\overline{M}_1$. We then compute lower bounds w_1 and $\overline{w}_1$ for the lengths of the circuits in M_1 and $\overline{M}_1$, respectively. We replace M_0 by the subset having the smaller lower bound and repeat the process, choosing a new edge e_2. These decisions can be represented by a tree (FIGURE 2 shows the case $w_1 < \overline{w}_1$).

Since the lower bounds may increase as we proceed down some branch of the tree, it may become profitable to return to a previously abandoned node and branch from there (e.g., return to $\overline{M}_1$ if $\overline{w}_1$ is less than both w_2 and $\overline{w}_2$). Continuing in this way, we eventually arrive at a node representing a single circuit of minimal length, i.e., a solution to the **TSP** for our network. There are several versions of this basic algorithm that differ in how the edges are chosen and the lower bounds are computed (see [**2**], [**23**] for a fuller discussion).

Of course, the same procedure will in principle work for the **TAP** if we compute lower bounds for scores rather than lengths. By (8) and (11),

$$L(P) + 2\,\mathrm{tr}\,S - (m+3)n \leqslant R(P)$$

for circuits derived from incidence matrices, so any lower bound for the lengths of all circuits in some set yields a lower bound for their scores. Thus it is easy to adapt any particular branch and bound algorithm for the **TSP** so that it also solves the **TAP**. This idea is due to Laporte [**20**].

Other solutions and models

In an application to genetics, Fulkerson and Gross [**7**] characterize pre-Petrie matrices in terms of interval graphs and give an algorithm for solving problem (**I**). Tucker [**29**] characterizes pre-Petrie matrices as incidence matrices having no submatrices of a certain form. In his survey of the field, Marquardt [**22**] discusses several algorithms for solving (**II**) and (**II**$'$). Kendall [**17**] discusses a very different model for seriation, based on the Shepard-Kruskal algorithm for multidimensional scaling. Kendall [**14**] and Wilkinson [**31**] present maximum-likelihood models. Finally, Laporte [18] also reports a solution using dynamic programming.

This work was supported in part by an NSF Science Faculty Professional Development Grant at the M.I.T. Operations Research Center. I wish to thank NSF for its support, MIT for its hospitality, and Gilbert Laporte for providing helpful references.

References

[1] J. J. Bartholdi III, J. B. Orlin, and H. D. Ratliff, Cyclic scheduling via integer programs with circular ones, Oper. Res., 28 (1980) 1074–1085.

[2] M. Bellmore and G. L. Nemhauser, The travelling salesman problem: a survey, Oper. Res., 16 (1968) 538–558.

[3] M. Borillo, W. Fernandez de la Vega, and A. Guenoche, eds., Raisonnement et Méthodes Mathématiques en Archéologie, Editions du CNRS, Paris, 1977.

[4] G. W. Brainerd, The place of chronological ordering in archaeological analysis, Amer. Antiquity, 16 (1951) 301–313.

[5] J. E. Doran, Computer analysis of data from the La Tène cemetery at Münsingen-Rain, in reference **[11]** below, pp. 422–431.

[6] J. E. Doran and F. R. Hodson, Mathematics and Computers in Archaeology, Harvard Univ. Press, Cambridge, 1975.

[7] D. R. Fulkerson and O. A. Gross, Incidence matrices and interval graphs, Pacific J. Math., 15 (1965) 835–855.

[8] M. R. Garey and D. S. Johnson, Computers and Intractability, Freeman, San Francisco, 1979.

[9] M. Grötschel, On the symmetrical travelling salesman problem: solution of 120-city problem, Mathematical Programming Study No. 12, North-Holland, Amsterdam, 1980, pp. 61–71.

[10] F. R. Hodson, The La Tène Cemetery at Münsingen-Rain, Stämpfli, Bern, 1968.

[11] F. R. Hodson, D. G. Kendall, and P. Tautu, eds., Mathematics in the Archaeological and Historical Sciences, Edinburgh Univ. Press, Edinburgh, 1971.

[12] L. Hubert, Some applications of graph theory and related non-metric techniques to problems of approximate seriation: the case of symmetric proximity measures, Br. J. Math. Stat. Psych., 27 (1974) 133–153.

[13] J. B. Kadane, Chronological ordering of archaeological deposits by the minimum path length method, Tech. Rep. No. 58, Dept. of Stat., Carnegie-Mellon University, 1972.

[14] D. G. Kendall, A statistical approach to Flinders Petrie's sequence-dating, Bull. Intern. Stat. Inst., 40 (1963) 657–681.

[15] ______, Incidence matrices, interval graphs and seriation in archaeology, Pacific J. Math., 28 (1969) 565–570.

[16] ______, Abundance matrices and seriation in archaeology, Z. Wahrsch. Verw. Gebiete, 17 (1971) 104–112.

[17] ______, Seriation from abundance matrices, in reference **[11]** above, pp. 215–252.

[18] G. Laporte, Permutation programming: problems, methods and applications, Pub. No. 197, Dépt. D'Informatique, Univ. de Montréal, 1975.

[19] ______, A comparison of two norms in archaeological seriation, J. Archaeol. Sci., 3 (1976) 249–255.

[20] ______, The seriation problem and the travelling salesman problem, J. Comput. Appl. Math., 4 (1978) 259–268.

[21] S. Lin and B. Kernighan, An effective heuristic algorithm for the traveling salesman problem, Oper. Res., 21 (1973) 498–516.

[22] W. H. Marquardt, Advances in archaeological seriation, in Advances in Archaeological Method and Theory, vol. 1 (M. B. Schiffer, ed.), Acad. Press, New York, 1978, pp. 266–314.

[23] P. Miliotis, Integer programming approaches to the travelling salesman problem, Math. Programming, 10 (1976) 367–378.

[24] M. W. Padberg and S. Hong, On the symmetrical travelling salesman problem: a computational study, Math. Programming Stud. No. 12, North-Holland, Amsterdam, 1980, pp. 78–107.

[25] C. Papadimitriou and K. Steiglitz, Combinatorial Optimization: Algorithm and Complexity, Prentice-Hall, 1982, pp. 410–418, pp. 455ff.

[26] W. M. F. Petrie, Sequences in prehistoric remains, J. Anthrop. Inst., N.S., 29 (1899) 295–301.

[27] F. S. Roberts, Discrete Mathematical Models, Prentice-Hall, Englewood Cliffs, 1976.

[28] W. S. Robinson, A method for chronologically ordering archaeological deposits, Amer. Antiquity, 16 (1951) 293–301.

[29] A. Tucker, A structure theorem for the consecutive 1's property, J. Combin. Theory Ser. B, 12 (1972) 153–162.

[30] E. M. Wilkinson, Archaeological seriation and the travelling salesman problem, in reference **[11]** above, pp. 276–283.

[31] ______, Techniques of data analysis: seriation theory, Archaeo-Physika: Technische und Naturwissenschaftliche Beiträge zur Feldarchäologie, 5 (1974) 1–142.

The Generalized Vandermonde Matrix

DAN KALMAN
University of Wisconsin-Green Bay
Green Bay, WI 54302

When you studied linear algebra, you probably encountered the identity

$$\det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1{}^2 & x_2{}^2 & x_3{}^2 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

in an example or an exercise. In addition, there is a good chance that a footnote informed you that this determinant is named for Vandermonde. In general, the matrix

$$V(x_1, x_2, \ldots, x_n) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

is called a **Vandermonde matrix**, and its determinant is given by the compact formula

$$\det V(x_1, \ldots, x_n) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i). \tag{1}$$

Have you ever wondered if there is any more to Vandermonde's matrix? Well of course there is!

For example, Vandermonde matrices arise when matrix methods are used in problems of polynomial interpolation, in solving differential equations, and in the analysis of recursively defined sequences. Yet, in each of these settings, the Vandermonde matrix tells only part of the story. For the rest of the story, a generalization of the Vandermonde matrix is required. In this note, I will discuss the three problem areas mentioned above and the role of the Vandermonde matrix in each. Then I will present what I call the **generalized Vandermonde matrix**, and discuss some of its properties. Throughout, where no confusion will result, $V(x_1, \ldots, x_n)$ will be shortened to V.

Polynomial interpolation

The process of fitting an $n-1$ degree polynomial to the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is usually referred to as **interpolation.** If the polynomial is

$$q(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1},$$

then the coefficients c_i may be determined by solving simultaneously the equations $q(x_j) = y_j$, $j = 1, 2, \ldots, n$. The matrix form for this system of equations is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{2}$$

Observe that the system matrix is a transposed Vandermonde matrix and that its determinant is connected with the solvability of the system. By identity (1), the determinant is nonzero if the values x_i are distinct, in which case the coefficients of q are uniquely determined. In fact, $q(x)$ can be explicitly formulated in this case as follows. Let

$$Q(x) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & x \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & x^{n-1} \\ y_1 & y_2 & \cdots & y_n & 0 \end{bmatrix}.$$

Then $Q(x_i)$ may be found by replacing x by x_i in the last column of the matrix. Subtracting column i from the last column now produces entries of 0, except for the last entry which is $-y_i$. Consequently, $Q(x_i) = (-\det V)y_i$, and so $q(x) = \frac{-1}{\det V} Q(x)$. Here, the Vandermonde determinant is of obvious significance. But what if the x_i are not distinct? Then $Q(x_i) = 0$ for all i, and system (2) has a singular matrix. Moreover, the polynomial interpolation problem is not meaningful if the x_i are not distinct. A more general kind of interpolation is meaningful, though, and as we shall see, involves the generalized Vandermonde matrix.

Differential equation initial value problems

Consider the differential equation

$$D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 Dy + a_0 y = 0, \tag{3}$$

where D represents differentiation with respect to t, and the initial conditions

$$D^j y(0) = y_j \quad j = 0, 1, 2, \ldots, n-1. \tag{4}$$

Together, these conditions comprise an initial value problem. To solve (3), it is customary to determine a factorization $(D - x_1)(D - x_2) \cdots (D - x_n)$ of the **characteristic polynomial** $p(D) = D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$. Then the functions $y = e^{x_i t}$, and all their linear combinations, satisfy (3) (see [**10**, p. 877]). In particular, we seek a linear combination

$$y = c_1 e^{x_1 t} + c_2 e^{x_2 t} + \cdots + c_n e^{x_n t}$$

which will satisfy (4). The coefficients c_i must be chosen so that

$$c_1 x_1^j + c_2 x_2^j + \cdots + c_n x_n^j = y_j, \quad j = 0, 1, 2, \ldots n-1.$$

If $C = [c_1 c_2 \ldots c_n]^T$ and $Y = [y_0 y_1 \ldots y_{n-1}]^T$, then this system can be written in matrix notation as

$$VC = Y. \tag{5}$$

If the x_i are distinct, (1) shows that (5) is uniquely solvable. And what if the x_i are not distinct? As in the previous situation, a generalization of the Vandermonde matrix may be used to complete the analysis.

Recursively defined sequences

Let the first n terms, $y_0, y_1, \ldots, y_{n-1}$, of a sequence be given. In many instances, the next term, and each term thereafter, is recursively defined as a linear combination of its first n predecessors by the equation

$$y_{k+n} = -a_{n-1} y_{k+n-1} - a_{n-2} y_{k+n-2} - \cdots - a_0 y_k,$$

where the values of a_i are not dependent on k. (The minus signs are included to simplify notation below.) A famous example of this situation is the **Fibonacci sequence** which begins $0, 1, 1, 2, 3, \ldots$ and where each term is the sum of the two terms which precede it. Viewed in a slightly different way, we may specify a sequence $\{y_j\}$ by requiring that any $n+1$ consecutive terms satisfy an equation of the type

$$y_{k+n}+a_{n-1}y_{k+n-1}+\cdots+a_1y_{k+1}+a_0y_k=0 \tag{6}$$

where the initial values $y_0, y_1, y_2, \ldots, y_{n-1}$ are specified. Equations of form (6) are called **difference equations** and are an important tool in modeling many systems of interest in the social sciences ([**4**], [**8**], [**9**]).

The difference equation (6), together with a set of initial values, forms an **initial value problem**. It is evident that a unique solution $\{y_j\}$ exists, for we may generate the terms of the sequence inductively. There is a method for "finding" this solution, that is, expressing y_j as a function of j, that parallels the method discussed above for differential equations.

We define an operator L (sometimes called the lag operator) which shifts the sequence $y_0, y_1, y_2, \ldots$ to the left, producing $y_1, y_2, y_3, \ldots$, and symbolize this with the notation $L\{y_j\}=\{y_{j+1}\}$. Equation (6) now can be written

$$L^n\{y_j\}+a_{n-1}L^{n-1}\{y_j\}+\cdots+a_1L\{y_j\}+a_0\{y_j\}=\{0\}. \tag{7}$$

There is again a characteristic polynomial $p(L)=L^n+a_{n-1}L^{n-1}+\cdots+a_0$ which is factored as $(L-x_1)(L-x_2)\cdots(L-x_n)$. The geometric sequences $\{x_i^j\}_{j=0}^{\infty}$ determined by the roots x_i, and all linear combinations of these sequences, satisfy (7) (see [**4**, p. 163]). A linear combination specified by

$$y_j=c_1x_1^j+c_2x_2^j+\cdots+c_nx_n^j$$

solves the initial value problem provided that the coefficients c_j satisfy (5). Once again, the Vandermonde determinant governs the solvability of the problem; once again, equation (1) assures a solution if the x_i are distinct; and once again, some further analysis is required if the x_i are not distinct.

In each case above, the Vandermonde matrix arises in connection with the problem of choosing coefficients in a linear combination. In the polynomial interpolation problem, it is possible to formulate the solution in a way that involves V directly and avoids the linear combination approach. This is also possible in the differential equation and difference equation problems. By developing this direct approach, we will be led to a more general problem, and to the generalized Vandermonde matrix.

The differential equation (3) may be transformed into a system of first order equations by introducing n variables, $y_1, y_2, \ldots, y_n$, and requiring $y_1=y$, $y_2=Dy_1$, $y_3=Dy_2$, and so on. Then equation (3) becomes

$$\begin{bmatrix} Dy_1\\ Dy_2\\ \vdots\\ Dy_{n-1}\\ Dy_n\end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix}\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_{n-1}\\ y_n\end{bmatrix}. \tag{8}$$

This last equation is written more compactly as

$$DY=AY \tag{9}$$

where Y is the vector with components y_i and A is the $n\times n$ matrix on the right side of (8). Incorporating the initial values $y_0, y_1, \ldots, y_{n-1}$ in a column matrix Y_0, the initial conditions (4) become

$$Y(0)=Y_0. \tag{10}$$

Finally, the unique solution to (9) and (10) is given by $Y(t)=e^{At}\cdot Y_0$ (see [**2**, p. 200]).

Now the Vandermonde matrix re-enters the picture, along with the roots $x_1, x_2, \ldots, x_n$ of the characteristic polynomial p. If the roots are distinct, V is invertible, and it can be shown that

$$A=V\,\mathrm{Diag}[x_1,\ldots,x_n]V^{-1} \tag{11}$$

(where the middle factor is a diagonal matrix with the specified entries). Similarly, the exponential matrix e^{At} simplifies to $V\text{Diag}[e^{x_1 t},\ldots,e^{x_n t}]V^{-1}$. The solution to the initial value problem specified by (9) and (10) is then given by

$$Y = V\text{Diag}\left[e^{x_1 t},\ldots,e^{x_n t}\right]V^{-1}Y_0. \tag{12}$$

Note that the product $V^{-1}Y_0$ is the solution for (5).

The difference equation (7) is transformed into a system of equations in a similar fashion. We consider a sequence of vectors $\{Y_j\}$ the components of which are the sequences $\{y_j\}$, $L\{y_j\}$, $L^2\{y_j\}$, etc. Equation (7) becomes

$$L\{Y_j\} = \{AY_j\} \tag{13}$$

with matrix A defined as before. Since $L\{Y_j\} = \{Y_{j+1}\}$, equation (13) is equivalent to $Y_{j+1} = AY_j$ and it is evident that $Y_j = A^j Y_0$ must follow. Applying (11) now produces

$$Y_j = V\text{Diag}\left(x_1^j,\ldots,x_n^j\right)V^{-1}Y_0$$

as the solution to the difference equation initial value problem.

The connection between the differential and difference equations can now be perceived quite clearly. Each can be formulated as a matrix equation involving A. The solution to each involves a function of A which is calculated simply when A can be reduced to diagonal form. The matrix V appears in precisely the case when A can be diagonalized. The question "What if the x_i are not distinct?" now becomes "What if A can't be diagonalized?"

The problem of diagonalizing a matrix M is bound up with eigenvalues. In general, the polynomial $p(x) = \det(M - xI)$ is called the **characteristic polynomial** for M; the roots of p are the eigenvalues of M. A sufficient condition for M to be diagonalizable is that p has n distinct roots, and these roots appear in the diagonal matrix. In the case we have been studying, $M = A$ and the characteristic polynomial p coincides with the previous bearers of that title. In particular, the coefficients of p appear in the bottom row of A. What is more (for this particular matrix), A is diagonalizable if and only if A has n distinct eigenvalues. For these reasons, A is sometimes called the **companion matrix** of p.

If p has some multiple roots, A cannot be diagonalized, but it can be reduced to a very simple form. The roots of p are arranged on the diagonal of a matrix J so that equal roots are adjacent. For each block of equal roots that appears, a string of 1's occurs on the superdiagonal. All other entries of J are zero. The matrix J is called the Jordan form of A. The problem of reducing a matrix to its Jordan form, of which diagonalization is a special case, is detailed in nearly every theoretical linear algebra text (see [**5**] for one example). The matrix that takes the role of V in reducing A to its Jordan form is what I call the **generalized Vandermonde matrix.**

For the remainder of the discussion, let

$$p(x) = (x - x_1)^{n_1}(x - x_2)^{n_2}\cdots(x - x_m)^{n_m},$$

where the roots x_i are distinct, and the positive multiplicities n_i sum to n. The generalized Vandermonde matrix $V(p)$ is defined as follows. Denoting by $f(t)$ the column $[1 t \cdots t^{n-1}]^T$ and by $f^{(k)}(t)$ the kth derivative of this column, define R_i as the $n \times n_i$ matrix with columns $\frac{1}{k!}f^{(k)}(x_i)$ for $k = 0, 1, \ldots, n_i - 1$. The **generalized Vandermonde matrix** is

$$V(p) = [R_1 R_2 R_3 \cdots R_m].$$

For example, if $p(x) = (x-a)^3(x-b)^2$, then $V(p)$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ a & 1 & 0 & b & 1 \\ a^2 & 2a & 1 & b^2 & 2b \\ a^3 & 3a^2 & 3a & b^3 & 3b^2 \\ a^4 & 4a^3 & 6a^2 & b^4 & 4b^3 \end{bmatrix}.$$

It should be noted that the use of differentiation in the definition of $V(p)$ is a notational convenience. One may equally well define the ij entry of R_k to be $\binom{i-1}{i-j} x_k^{i-j}$ for $i \geqslant j$ and zero otherwise. Note also that $V(p) = V$, the usual Vandermonde matrix, if p has n distinct roots. The generalized Vandermonde matrix has appeared previously as a special kind of confluent alternant matrix. A general discussion of alternants and confluent alternants may be found in Chapter VI of [**1**].

How does $V(p)$ complete the analysis in the problems already studied? First, its determinant is given by

$$\det V(p) = \prod_{1 \leqslant i < j \leqslant m} (x_j - x_i)^{n_i n_j} \tag{14}$$

which is a generalization of (1). In particular, $V(p)$ is invertible.

As stated earlier, $V(p)$ appears in the reduction of A to its Jordan form J, specifically, $A = V(p)JV(p)^{-1}$. Thus, in the differential equation initial value problem, the solution may generally be written

$$Y(t) = V(p)e^{Jt}V(p)^{-1}Y_0,$$

and for the difference equation the solution is

$$Y_j = V(p)J^jV(p)^{-1}Y_0.$$

In each case, the solution previously studied is recovered when J is diagonal and $V(p) = V$. When J is not diagonal, e^{Jt} and J^j are more complicated than in the diagonal case, but are still simpler than e^{At} and A^j, respectively.

As a generalization of polynomial interpolation we have osculatory interpolation where values are specified for a polynomial q and its first $n_i - 1$ derivatives at x_i. These conditions may be formulated as

$$q^{(j)}(x_i) = j!y_{ij} \quad 1 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant n_i - 1.$$

The polynomial q is generally different from p, which specifies the locations of x_i and the number of derivatives which have values prescribed for q.

As in regular polynomial interpolation, it is possible to solve a system of equations to determine the coefficients for q, and the system matrix is $V(p)^T$. Alternatively, let

$$Q(x) = \det \left[\begin{array}{c|c} & 1 \\ & x \\ V(p) & x^2 \\ & \vdots \\ & x^{n-1} \\ \hline y_{10} \; y_{11} \cdots y_{m,n_m-1} & 0 \end{array} \right].$$

Using methods similar to those employed in the discussion of regular interpolation, it is easy to show that $q(x) = -\dfrac{1}{\det V(p)} Q(x)$. This solution reduces to the previous one when $V(p) = V$.

In all of these problems, to use $V(p)$ we must know that $\det V(p) \neq 0$. This is guaranteed by equation (14), which is remarkable in its own right as an extension of (1). I conclude by providing an overview of several approaches to proving (14), with an aim to illuminate some of the properties of $V(p)$. An additional approach utilizing row operations is suggested in [**1**, p. 120].

One approach to establishing (14) involves repeated differentiation of (1). The product is differentiated as a polynomial in x_i while the derivative of the determinant with respect to x_i is computed by differentiating the ith column. Thus, if $p(x)=(x-a)^3(x-b)^2$, we compute

$$\frac{1}{2}\left(\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_3}\right)^2\left(\frac{\partial}{\partial x_5}\right)\det V(x_1,x_2,x_3,x_4,x_5)=\det\begin{bmatrix}1 & 0 & 0 & 1 & 0\\ x_1 & 1 & 0 & x_4 & 1\\ x_1^2 & 2x_2 & 1 & x_4^2 & 2x_5\\ x_1^3 & 3x_2^2 & 3x_3 & x_4^3 & 3x_5^2\\ x_1^4 & 4x_2^3 & 6x_3 & x_4^4 & 4x_5^3\end{bmatrix}$$

and then replace x_1, x_2, x_3 by a and x_4, x_5 by b to form $\det V(p)$. In principle, these same operations could be carried out on the right side of (1) to yield

$$\det V(p)=\frac{1}{2}\left(\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_3}\right)^2\left(\frac{\partial}{\partial x_5}\right)\prod_{1\leqslant i<j\leqslant 5}(x_j-x_i)\Bigg|_{\substack{x_1=x_2=x_3=a\\ x_4=x_5=b}}$$

However, the differentiation at right is cumbersome, making it difficult to verify that $V(p)=(b-a)^6$ as given by (14).

Alternatively, we might proceed by induction on the sum of exponents n_i with $n_i>1$. When this sum is zero (i.e., all $n_i=1$), the generalized Vandermonde matrix becomes a regular Vandermonde matrix and (14) reduces to (1) which is known to hold. For the example $p(x)=(x-a)^3(x-b)^2$, the sum of exponents which exceed 1 is $3+2=5$. To illustrate the induction step, let

$$p_1(x)=(x-a)^2(x-t)(x-b)^2$$

so that the sum of exponents greater than 1 is 4, and

$$V(p_1)=\begin{bmatrix}1 & 0 & 1 & 1 & 0\\ a & 1 & t & b & 1\\ a^2 & 2a & t^2 & b^2 & 2b\\ a^3 & 3a^2 & t^3 & b^3 & 3b^2\\ a^4 & 4a^3 & t^4 & b^4 & 4b^3\end{bmatrix}. \tag{15}$$

Assuming (14) for p_1 yields

$$\det V(p_1)=(t-a)^2(b-t)^2(b-a)^4.$$

We can now prove that (14) holds for p, as well, by noting

$$\det V(p)=\frac{1}{2}\left(\frac{d}{dt}\right)^2\det V(p_1)|_{t=a}$$

and differentiating columns as before. In general, the polynomial $p_1(x)=\dfrac{(x-t)}{(x-x_i)}p(x)$ may be formed as long as at least one n_i exceeds 1. The differentiation is not as cumbersome as in the previous method of proof, and the induction step is manageable.

Another approach to the induction step avoids differentiation entirely. Replace t with $a+h$ and expand the powers of $a+h$ in the third column of $V(p_1)$, obtaining

$$\begin{bmatrix}1\\ a+h\\ (a+h)^2\\ (a+h)^3\\ (a+h)^4\end{bmatrix}=\begin{bmatrix}1\\ a\\ a^2\\ a^3\\ a^4\end{bmatrix}+h\begin{bmatrix}0\\ 1\\ 2a\\ 3a^2\\ 4a^3\end{bmatrix}+h^2\begin{bmatrix}0\\ 0\\ 1\\ 3a\\ 6a^2\end{bmatrix}+h^3\begin{bmatrix}0\\ 0\\ 0\\ 1\\ 4a\end{bmatrix}+h^4\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 1\end{bmatrix}.$$

Now we may compute $\det V(p_1)$ as a sum of the five determinants corresponding to the terms in the expansion above. The first two determinants vanish, resulting in

$$\det V(p_1) = h^2 \det V(p) + h^3 r(h),$$

where $r(h)$ is a polynomial in h. After rearrangement, this becomes

$$\det V(p) = \frac{1}{h^2}\det V(p_1) - hr(h).$$

Since $\det V(p_1) = h^2(b-a-h)^2(b-a)^4$ by the induction hypothesis, we now obtain

$$\det V(p) = (b-a-h)^2(b-a)^4 - hr(h),$$

and letting h go to zero provides the desired conclusion. In general, this approach leads to a formula of the form

$$\det V(p) = \lim_{h \to 0}\left(\frac{1}{h^{n_i - 1}}\det V(p_1) - hr(h)\right),$$

where $h = t - x_i$. Here, one can almost see the induction take place, for as h approaches zero, t coalesces with x_i and the sum of multiplicities greater than 1 moves up a notch. When $n_i = 2$, this process is identical to differentiation with respect to t followed by evaluation at $t = x_i$.

The last method of proof of (14) offered is inspired by a common proof of (1), which goes as follows. Let $g(t) = \det V(x_1, \ldots, x_{n-1}, t)$ and observe that g is a polynomial of degree $n-1$ with roots $x_1, \ldots, x_{n-1}$. The leading coefficient, obtained by expansion by minors in column n, is $\det V(x_1, x_2, \ldots, x_{n-1})$. Thus

$$g(t) = \det V(x_1, \ldots, x_{n-1}) \prod_{i=1}^{n-1} (t - x_i)$$

and (1) follows by induction on n.

The generalization of this method to the proof of (14) may be illustrated by considering again the example $p(x) = (x-a)^3(x-b)^2$. Here, take $g(t) = \det V(p_1)$, where $V(p_1)$ is shown in (15). As before, $g(t)$ has roots at $t = a$ and $t = b$. In addition, differentiation of the third column of $V(p_1)$ shows that $g'(t)$ also has roots at $t = a$ and $t = b$, so that a and b are repeated roots of $g(t)$. Now four roots of $g(t)$ are accounted for, which leads to

$$g(t) = C(t-a)^2(t-b)^2.$$

The leading coefficient, C, is the (5,3) cofactor of $V(p_1)$, and hence the determinant of a 4×4 generalized Vandermonde matrix. Using induction on n, the dimension of $V(p)$, let (14) be assumed for the computation of C. Finally, since $\frac{1}{2}g''(a) = \det V(p)$, to obtain the equivalent of (14) for p we calculate $\frac{1}{2}C\left(\frac{d}{dt}\right)^2[(t-a)^2(t-b)^2]$, substitute a for t, and simplify. Note that in this method of proof, the induction step involves reducing the degree of p and the size of $V(p)$, while in the previous methods, the multiplicity of a single root of p is reduced while maintaining the degree and the size of the matrix.

References

[1] A. C. Aitken, Determinants and Matrices, 9th ed., Interscience Pub., New York, 1956.

[2] Tom M. Apostol, Calculus, vol. 2, 2nd ed., Blaisdell, Waltham, Mass., 1969.

[3] William Snow Burnside and Arthur William Panton, The Theory of Equations, vol. 2, Dover, New York, 1960.

[4] Samuel Goldberg, Introduction to Difference Equations, Wiley, New York, 1967.

[5] Kenneth Hoffman and Ray Kunze, Linear Algebra, Prentice-Hall, Englewood Cliffs, N.J., 1961.

[6] Kenneth S. Miller, Linear Difference Equations, Benjamin, New York, 1968.

[7] L. M. Milne-Thompson, The Calculus of Finite Differences, St. Martin's Press, New York, 1960.

[8] Barbara Salert, Public Support for Presidents I & II, Birkhauser, Boston, Cambridge, Mass., 1980, UMAP units 299–300.

[9] Gilbert Strang, Linear Algebra and Its Applications, Academic Press, New York, 1976.

[10] George B. Thomas, Jr., Calculus and Analytic Geometry, 5th ed., Addison-Wesley, Reading, Mass. 1979.

The Group of Primitive Pythagorean Triangles

ERNEST J. ECKERT
Aalborg University Center
9000 Aalborg, Denmark

A famous identity shows that *the product of two integers, each of which is a sum of two squares, is again a sum of two squares*:

$$(a^2+b^2)(A^2+B^2)=(aA-bB)^2+(aB+bA)^2. \tag{1}$$

If $a^2+b^2=c^2$ and $A^2+B^2=C^2$, with c and C integers, then (1) suggests a way of constructing a pythagorean triangle from two given ones ([**1**], p. 488). If we let (a,b,c) denote the pythagorean triangle with sides a, b and hypotenuse c, we may define an operation (addition) which associates to two pythagorean triangles (a,b,c) and (A,B,C), another pythagorean triangle:

$$(a,b,c)+(A,B,C)=(aA-bB,aB+bA,cC). \tag{2}$$

For the moment we assume that $aA-bB>0$. Examples of the addition defined in (2) are:

$$(4,3,5)+(12,5,13)=(4\cdot 12-3\cdot 5,4\cdot 5+3\cdot 12,5\cdot 13)=(33,56,65),$$
$$(4,3,5)+(15,8,17)=(36,77,85),$$
$$(4,3,5)+(4,3,5)=2(4,3,5)=(7,24,25).$$

It is known—though perhaps not well known—that this addition of pythagorean triangles can be made into a group operation when certain conventions (illustrated below) are adhered to (see [**8**]). In this note, we shall investigate the structure of this group. In addition, we will find that in solving certain equations in the group we are led, in a natural way, to answer a question posed by W. Sierpinski ([**7**], p. 32): *How many (primitive) pythagorean triangles have the same hypotenuse*?

A pythagorean triangle (a,b,c) is either **primitive** (i.e., a, b, c have no common factor), or it is a multiple of a primitive one, (ka,kb,kc). Rather than consider the set of all pythagorean triangles, we consider only the set **P** of *primitive* pythagorean triangles, identifying each pythagorean triangle (x,y,z) with its primitive ancestor (e.g., $(8,6,10)$ is identified with $(4,3,5)$). We show that this set, with the operation in (2), is a group. An identification of the triples (a,b,c) in **P** with points on the unit circle in the complex plane gives a useful geometric interpretation. Each primitive pythagorean triangle (a,b,c) determines a complex number $z=a+ib$, with $|z|=\sqrt{a^2+b^2}=c$. The line from the origin to z cuts the unit circle in the complex plane (FIGURE 1) at the point

$$e^{i\alpha}=\frac{a}{c}+i\frac{b}{c}=\cos\alpha+i\sin\alpha. \tag{3}$$

We identify (a,b,c) with the point $e^{i\alpha}$ in (3). The product of two such numbers $e^{i\alpha}$, $e^{i\beta}$ is

$$e^{i\alpha}e^{i\beta}=e^{i(\alpha+\beta)}=\left(\frac{a}{c}+i\frac{b}{c}\right)\left(\frac{A}{C}+i\frac{B}{C}\right)=\frac{aA-bB}{cC}+i\frac{aB+bA}{cC}. \tag{4}$$

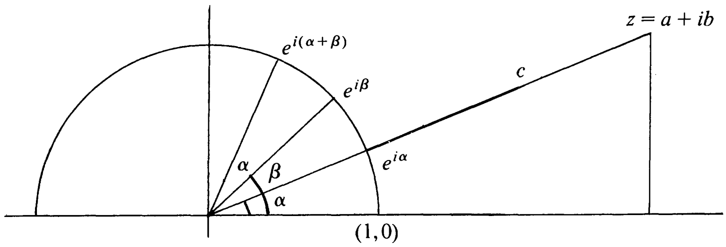


FIGURE 1

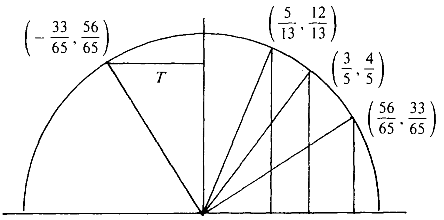


FIGURE 2

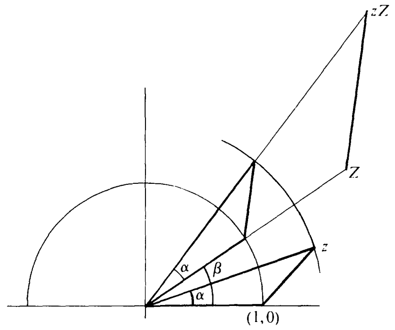


FIGURE 3

Thus the complex product corresponds to the pythagorean triangle $(aA - bB, aB + bA, cC)$, which is exactly the sum of (a, b, c) and (A, B, C) defined in (2).

Since our addition of pythagorean triangles corresponds to addition of angles, the operation defined in (2) is associative and commutative. This suggests that the identity element of our group should be the degenerate pythagorean triangle $(1,0,1)$ with zero angle. And it works:

$$(1,0,1) + (a, b, c) = (1a - 0b, 1b + 0a, 1c) = (a, b, c).$$

To insure that **P** is closed under our operation, we need to take care of the case ignored thus far: How do we modify (2) when $aA - bB \leqslant 0$, i.e., the angle sum $\gamma = \alpha + \beta \geqslant \pi/2$. For example, (2) defines $(3,4,5) + (5,12,13) = (-33,56,65)$ and $(a, b, c) + (b, a, c) = (0, c^2, c^2) = (0,1,1)$. The triangle T (FIGURE 2) in the second quadrant defined by $(\cos\gamma, \sin\gamma) = (-33/65, 56/65)$ is congruent to the triangle in the first quadrant defined by $(56/65, 33/65)$ obtained by rotating T about the origin through an angle of $-\pi/2$. Accordingly, we modify (4) so that whenever $\gamma = \alpha + \beta \geqslant \pi/2$ we reduce the angle sum modulo $\pi/2$. In the other example, if we reduce the angle $\gamma = \pi/2$ modulo $\pi/2$ we get 0, which corresponds to the zero element $(1,0,1)$. Hence, our second example shows that (a, b, c) and (b, a, c) are inverses of each other.

Summarizing, *the set* **P** *of primitive pythagorean triangles is a group under the operation, called addition, defined by*

$$(a, b, c) + (A, B, C) = \begin{cases} (aA - bB, bA + aB, cC) & \text{when } aA - bB > 0 \\ (bA + aB, bB - aA, cC) & \text{when } aA - bB \leqslant 0. \end{cases} \tag{5}$$

The identity element in **P** *is* $(1,0,1)$, *and the inverse of* (a, b, c) *is* (b, a, c).

The geometric construction of the product of two complex numbers gives an alternate interpretation of the group operation (see [**4**], Vol. I, p. 22). Given (a, b, c) and (A, B, C) in **P**, we can multiply $z = a + ib$ and $Z = A + iB$ geometrically, as illustrated in FIGURE 3 for the case where $\alpha + \beta < \pi/2$. It is not hard to modify the construction to the cases $\alpha + \beta = \pi/2$ and $\alpha + \beta > \pi/2$. Thus, by identifying (a, b, c) with the gaussian integer $a + ib$, we can perform the addition of pythagorean triangles by a purely geometric construction and obtain the sum triangle directly.

Note that although (a, b, c) and its inverse (b, a, c) are congruent triangles, we have to distinguish between them as group elements in **P**. Denote by $\langle(a, b, c)\rangle$ the subgroup of **P** generated by the element (a, b, c); the members of this subgroup are the integral multiples of (a, b, c). For n positive, $n(a, b, c) = (a, b, c) + \cdots + (a, b, c)$, n summands, and $-n(a, b, c)$ is the inverse of $n(a, b, c)$ in **P**. The first five multiples of $(4,3,5)$ are: $(4,3,5)$, $(7,24,25)$, $(117,44,125)$, $(336,527,625)$ and $(3116,237,3125)$. Interchanging the two first coordinates in each of these, we obtain the first five negative multiples of $(4,3,5)$. The multiple $0 \cdot (4,3,5)$ is defined to be the identity element $(1,0,1)$. Since $\langle(a, b, c)\rangle = \langle(b, a, c)\rangle$ we may take either (a, b, c) or (b, a, c) as generator of the subgroup $\langle(a, b, c)\rangle$. We choose the one, say (a, b, c), for which the determining angle α is less than $\pi/4$, i.e., $a > b$.

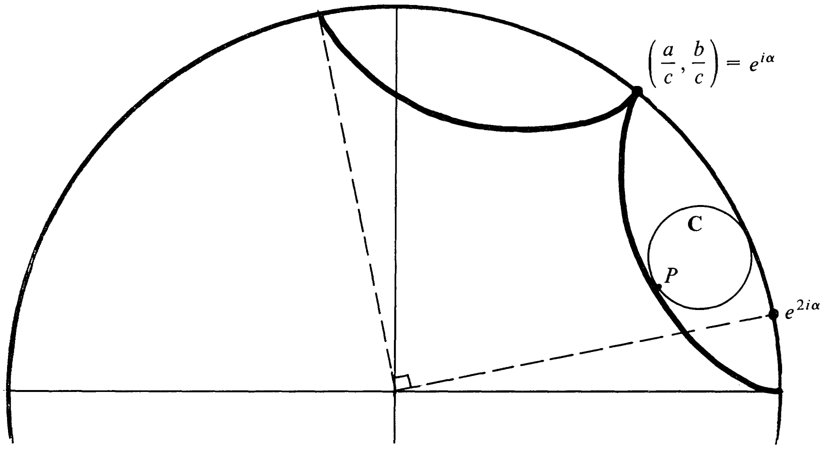


FIGURE 4

From the geometric interpretation of addition in **P**, we can show that *each primitive pythagorean triangle, except the degenerate one, generates an infinite cyclic subgroup of* **P**. Each nonzero element of **P** then is of infinite order, so that **P** is a torsionfree group.

If (a, b, c) is identified with $e^{i\alpha}$ on the unit circle U, then a circle **C** with radius $r = \alpha/2\pi$ is made to roll inside the unit circle U in the positive (counterclockwise) direction. The point P on **C** initially at $(1,0)$ will, as the circle **C** makes one complete revolution, mark off the point $(a/c, b/c) = e^{i\alpha}$ on U. As the circle **C** rolls on, the initial point P will mark off points on U which, after reduction modulo $\pi/2$, identify the increasing positive multiples of (a, b, c) (see FIGURE 4). Rolling the circle **C** in the negative direction, the marks on U will identify the negative multiples of (a, b, c), i.e., the positive multiples of the inverse element (b, a, c). The path traced out by the initial point P is a hypocycloid, and the points marked off on U are its cusps. A hypocycloid is a closed curve if and only if $\alpha/2\pi$ is rational. Since a rational multiple of π cannot have rational sine and cosine, $\alpha/2\pi$ must be irrational, ([**6**], p. 21). Our hypocycloid then, is not a closed curve, hence $\langle(a, b, c)\rangle$ is infinite. Another consequence of this observation is that the set of cusps forms a dense subset of U in the first quadrant. Thus, a cyclic subgroup generated by a single element provides a dense subset of points on U in the first quadrant. This is an improvement of the known result that the set of points $(a/c, b/c)$ on U corresponding to *all* pythagorean triangles (a, b, c) is a dense subset of U in the first quadrant, ([7], pp. 89–91). See also problem 6053 in [3].

We return now to the investigation of the structure of the group **P**. From the fact that **P** is a torsion free group it is not farfetched to guess that **P** is a free abelian group, i.e., that **P** is a direct sum of infinite cyclic groups. We will prove this, i.e., we will show that if $\{g_i\}$, $g_i \in \mathbf{P}$, is the set of generators of the cyclic subgroups of **P**, then each $g \in \mathbf{P}$ is a unique finite linear combination

$$g = n_1 g_{i_1} + n_2 g_{i_2} + \cdots + n_k g_{i_k}$$

with different $g_{i_1}, \ldots, g_{i_k}$, the coefficients n_i being nonzero integers and k a positive integer. We first need a Lemma.

LEMMA. *If a, b, r, s are integers and p is an odd prime such that $rsab \not\equiv 0 \bmod p^2$ and $r^2 + s^2 \equiv a^2 + b^2 \equiv 0 \bmod p^2$, then exactly one of the following is true:*

$$ra + sb \equiv rb - sa \equiv 0 \bmod p^2,$$

or

$$ra - sb \equiv rb + sa \equiv 0 \bmod p^2.$$

Proof. We have

$$(ra+sb)(ra-sb)=r^2a^2-s^2b^2=r^2(a^2+b^2)-(r^2+s^2)b^2\equiv 0 \bmod p^2.$$

Now p cannot divide both factors on the left, since then p would divide $2ra$ and $2sb$, contrary to hypothesis. Hence, either $ra+sb\equiv 0 \bmod p^2$, or $ra-sb\equiv 0 \bmod p^2$. In the first case we write

$$(ra+sb)^2+(rb-sa)^2=(r^2+s^2)(a^2+b^2)\equiv 0 \bmod p^4,$$

and conclude that $rb-sa\equiv 0 \bmod p^2$. In the other case,

$$(ra-sb)^2+(rb+sa)^2=(r^2+s^2)(a^2+b^2)\equiv 0 \bmod p^4$$

implies $rb+sa\equiv 0 \bmod p^2$.

PROPOSITION. *The group* **P** *of primitive pythagorean triangles is a free abelian group which is generated by the set of triangles* (a,b,p) *with* p *prime*, $p\equiv 1 \bmod 4$, *and* $a>b$.

Proof. Let $(r,s,d)\in \mathbf{P}, (r,s,d)\neq(1,0,1)$, and let the prime decomposition of d be

$$d=p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}. \tag{6}$$

Our goal is to show that we can write

$$(r,s,d)=e_1n_1(a_1,b_1,p_1)+\cdots+e_kn_k(a_k,b_k,p_k) \tag{7}$$

where $a_j>b_j, e_j=\pm 1, p_j\equiv 1 \bmod 4$ for $j=1,2,\ldots,k$. Recall that a primitive pythagorean triangle (r,s,d), is uniquely determined by a pair of natural numbers $m,n(m>n)$ which are relatively prime and one of which is even:

$$r=m^2-n^2, s=2mn, d=m^2+n^2, \quad \text{or}$$
$$r=2mn, s=m^2-n^2, d=m^2+n^2.$$

A positive integer d can be a hypotenuse in a pythagorean triangle if and only if d has a prime factor of the form $4h+1$, ([7], p. 27). It follows that a prime p can be a hypotenuse if and only if $p\equiv 1 \bmod 4$. Such a prime has a unique representation as a sum of two squares: $p=m^2+n^2$, ([**5**], p. 128). With $a=m^2-n^2$, $b=2mn$, we have $p^2=a^2+b^2$. Thus, if in (6) $d=p$ (the case $k=1$, $n_1=1$), then we must have either

$$(r,s,d)=(a,b,p) \text{ or } (r,s,d)=(b,a,p)=-(a,b,p),$$

which is in agreement with (7). (Note that there are exactly two pythagorean triangles having p as hypotenuse: (a,b,p), (b,a,p). These are congruent as triangles, and inverses in the group **P**.)

Consider next the triangle (r,s,d) where $d=pq$, p and q primes, with $p\equiv 1 \bmod 4$. Let a,b be the unique pair for which $a^2+b^2=p^2$, $a>b$. The two equations $(r,s,pq)=(x,y,z)\pm(a,b,p)$ have, respectively, the solutions

$$(x,y,z)=(r,s,pq)-(a,b,p)=(rb-sa,sb+ra,p^2q)$$

and (8)

$$(x,y,z)=(r,s,pq)+(a,b,p)=(ra-sb,sa+rb,p^2q).$$

Since $rsab\not\equiv 0 \bmod p^2$ and $r^2+s^2\equiv a^2+b^2\equiv 0 \bmod p^2$, the Lemma applies, so that in exactly one of the solutions (8), p^2 is a common factor of all three coordinates in the triple on the right. Cancelling p^2 from this solution, we conclude that $(x,y,z)=(u,v,q)$ where either

$$u=\frac{rb-sa}{p^2}, v=\frac{sb+ra}{p^2} \text{ are integers,}$$

or

$$u=\frac{ra-sb}{p^2}, v=\frac{sa+rb}{p^2} \text{ are integers.}$$

This means that either

$$(r, s, pq) = (u, v, q) + (a, b, p),$$

or (9)

$$(r, s, pq) = (u, v, q) - (a, b, p).$$

We have, so to speak, peeled off p from $d = pq$. Suppose $p \neq q$. Since q is a hypotenuse in a pythagorean triangle, (u, v, q), it follows that $q \equiv 1 \bmod 4$. Now let A, B be the unique pair for which $A^2 + B^2 = q^2, A > B$; arguing as in the case $d = p$, we see that either $(u, v, q) = (A, B, q)$ or $(u, v, q) = -(A, B, q)$. Combining this with (9), we have that (r, s, d) must be exactly one of the following:

$$(A, B, q) + (a, b, p), \qquad (A, B, q) - (a, b, p),$$
$$-(A, B, q) + (a, b, p) \quad \text{or} \quad -(A, B, q) - (a, b, p),$$

which is in agreement with (7). (We note that there are four primitive pythagorean triangles with the same hypotenuse $d = pq$.) If $p = q$, then the solution (9) becomes

$$(r, s, d) = (u, v, p) + (a, b, p)$$

or

$$(r, s, d) = (u, v, p) - (a, b, p)$$

where u, v are certain integers. Since a, b are the unique integers satisfying $a^2 + b^2 = p^2$, it follows that for the first solution, $(u, v, p) = (a, b, p)$, since the other choice, $(u, v, p) = (b, a, p)$, would lead to $(r, s, d) = (1,0,1)$, which is not so. Similarly, for the other solution, $(u, v, p) = -(a, b, p)$, and so either $(r, s, p^2) = 2(a, b, p)$ or $(r, s, p^2) = -2(a, b, p)$, which is in agreement with (7). (Note that there are exactly two triangles having the same hypotenuse $d = p^2$, namely, $2(a, b, p)$ and $-2(a, b, p)$.)

The prescription for the general case, $d = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, is now clear: peel off one prime at a time. Since only a finite number of primes can occur, we will eventually arrive at (7). Note that inherent in the proof is the result that in a primitive pythagorean triangle the hypotenuse is divisible only by primes which are congruent to 1 modulo 4.

How many primitive pythagorean triangles with the same hypotenuse, $d = p_1^{n_1} \cdots p_k^{n_k}$, are there? Formula (7) says that any such triangle, (r, s, d), must be of the form

$$\pm n_1(a_1, b_1, p_1) \pm \cdots \pm n_k(a_k, b_k, p_k).$$

The answer, then, is just 2^k. For example, if $d = 125 = 5^3$ then $(r, s, d) = (r, s, 5^3) = \pm 3(4,3,5) = \pm(117,44,125)$ are the only two (2^1) possible. If $d = 5^2 13$ then there are $2^2 = 4$ triangles with hypotenuse $d = 325$. They are

$$2(4,3,5) + (12,5,13) = (323,36,325), \quad 2(4,3,5) - (12,5,13) = (204,253,325),$$
$$-2(4,3,5) + (12,5,13) = (253,204,325) \text{ and } -2(4,3,5) - (12,5,13) = (36,323,325).$$

Ch. L. Schedd [2] found 64 different triangles with hypotenuse $d = 2,576,450,045$. Since $d = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 \cdot 41 \cdot 53$, our count shows that there should be $2^7 = 128$. Schedd did not distinguish between (r, s, d) and (s, r, d) as we do, so his count includes them all.

I wish to thank the referees and editor for their helpful suggestions in preparing this note.

References

[1] R. Courant and H. Robbins, What is Mathematics?, Oxford Univ. Press, 1941.

[2] Curiosa. Scripta Mathematica, 15 (1949) 131–132.

[3] Raphael Finkelstein, proposer and J. C. Lagarias, solver, Problem 6053, Amer. Math. Monthly, 84 (1977) 493.

[4] E. Hille, Analytic Function Theory, vol. 1, Blaisdell, 1959.
[5] W. J. LeVeque, Topics in Number Theory, vol. 1, Addison-Wesley, 1956.
[6] Ivan Niven, Irrational Numbers, The Carus Mathematical Monographs, no. 11, MAA, 1956.
[7] Waclaw Sierpinski, Pythagorean Triangles, The Scripta Mathematica Studies, no. 9, Yeshiva Univ., New York, 1962.
[8] Olga Taussky, Sums of squares, Amer. Math. Monthly, 77 (1970) 805-830.

On Removing a Ball Without Disturbing the Others

ROBERT DAWSON
Dalhousie University
Halifax, Nova Scotia, Canada B3H 4H8

In the game of Jackstraws, each player tries in turn to remove an object from a pile without disturbing any other object. A related problem was posed in William Moser's *Research Problems in Discrete Geometry*, 1981 [**1**]:

Given a finite set of solid spheres, prove that at least one of them can be moved without disturbing the others.

The same problem was posed for star-shaped regions. (Recall that a solid body B is **star-shaped** if B contains a point x_0 such that the line segment joining x_0 to any other point of B lies entirely inside B.)

This note presents a proof that any collection of m balls in $\mathbb{R}^n$, intersecting at most in their boundaries, has not only one, but $\min(m, n+1)$ members that can be so moved; and we will consider generalizations to other types of bodies. It will be shown that the result does not lend itself to easy generalization.

We say that a body B in a collection of bodies is **movable** if there exists a continuous one-parameter family $f_t(B)$, $0 \leqslant t \leqslant 1$, of rigid motions of B such that $f_0(B) = B$, $f_1(B) \neq B$, and for each t, $f_t(B)$ and any other body in the collection intersect at most in their boundaries. If, for every d, there exists such a family of rigid motions, in which f_1 moves some point in B through a distance greater than or equal to d, we say that B is **movable to infinity**.

THEOREM 1. *In any finite collection of m balls in* $\mathbb{R}^n$, *intersecting at most in their boundaries, there exist at least* $\min(m, n+1)$ *elements, each of which is movable to infinity.*

In order to prove this, we shall require the following:

LEMMA. *A member of a collection of smooth convex bodies is movable by some translation if and only if the set of vectors normal to its bounding surface at points of contact with other bodies lies in some closed hemisphere.*

In our arguments, it will be convenient to identify a point x in $\mathbb{R}^n$ with the vector $\vec{x}$ from the origin to x; notation will make the context clear. The lemma is proved by considering the effect of a translation of a body on the points of contact of that body with other bodies. For a given body B_i, let p_{ij} be all the points of contact with other bodies B_j; and let $\vec{n}_{ij}$ be the unit vectors normal to the surface of B_i at p_{ij}. Then, if B_i is translated along a vector $\vec{d}$, the image of each p_{ij} will penetrate a distance $\vec{d} \cdot \vec{n}_{ij}$ into the body B_j. This results in intersection if and only if one of these inner products is positive. Suppose that it does not; then all $\vec{n}_{ij}$ lie on the closed hemisphere opposite $\vec{d}$. Conversely, if for some B_i, all $\vec{n}_{ij}$ lie on one closed hemisphere, the body B_i may be translated along the vector most remote from that hemisphere.

[4] E. Hille, Analytic Function Theory, vol. 1, Blaisdell, 1959.
[5] W. J. LeVeque, Topics in Number Theory, vol. 1, Addison-Wesley, 1956.
[6] Ivan Niven, Irrational Numbers, The Carus Mathematical Monographs, no. 11, MAA, 1956.
[7] Waclaw Sierpinski, Pythagorean Triangles, The Scripta Mathematica Studies, no. 9, Yeshiva Univ., New York, 1962.
[8] Olga Taussky, Sums of squares, Amer. Math. Monthly, 77 (1970) 805-830.

On Removing a Ball Without Disturbing the Others

ROBERT DAWSON
Dalhousie University
Halifax, Nova Scotia, Canada B3H 4H8

In the game of Jackstraws, each player tries in turn to remove an object from a pile without disturbing any other object. A related problem was posed in William Moser's *Research Problems in Discrete Geometry*, 1981 [**1**]:

Given a finite set of solid spheres, prove that at least one of them can be moved without disturbing the others.

The same problem was posed for star-shaped regions. (Recall that a solid body B is **star-shaped** if B contains a point x_0 such that the line segment joining x_0 to any other point of B lies entirely inside B.)

This note presents a proof that any collection of m balls in $\mathbb{R}^n$, intersecting at most in their boundaries, has not only one, but $\min(m, n+1)$ members that can be so moved; and we will consider generalizations to other types of bodies. It will be shown that the result does not lend itself to easy generalization.

We say that a body B in a collection of bodies is **movable** if there exists a continuous one-parameter family $f_t(B)$, $0 \leqslant t \leqslant 1$, of rigid motions of B such that $f_0(B) = B$, $f_1(B) \neq B$, and for each t, $f_t(B)$ and any other body in the collection intersect at most in their boundaries. If, for every d, there exists such a family of rigid motions, in which f_1 moves some point in B through a distance greater than or equal to d, we say that B is **movable to infinity**.

THEOREM 1. *In any finite collection of m balls in* $\mathbb{R}^n$, *intersecting at most in their boundaries, there exist at least* $\min(m, n+1)$ *elements, each of which is movable to infinity.*

In order to prove this, we shall require the following:

LEMMA. *A member of a collection of smooth convex bodies is movable by some translation if and only if the set of vectors normal to its bounding surface at points of contact with other bodies lies in some closed hemisphere.*

In our arguments, it will be convenient to identify a point x in $\mathbb{R}^n$ with the vector $\vec{x}$ from the origin to x; notation will make the context clear. The lemma is proved by considering the effect of a translation of a body on the points of contact of that body with other bodies. For a given body B_i, let p_{ij} be all the points of contact with other bodies B_j; and let $\vec{n}_{ij}$ be the unit vectors normal to the surface of B_i at p_{ij}. Then, if B_i is translated along a vector $\vec{d}$, the image of each p_{ij} will penetrate a distance $\vec{d} \cdot \vec{n}_{ij}$ into the body B_j. This results in intersection if and only if one of these inner products is positive. Suppose that it does not; then all $\vec{n}_{ij}$ lie on the closed hemisphere opposite $\vec{d}$. Conversely, if for some B_i, all $\vec{n}_{ij}$ lie on one closed hemisphere, the body B_i may be translated along the vector most remote from that hemisphere.

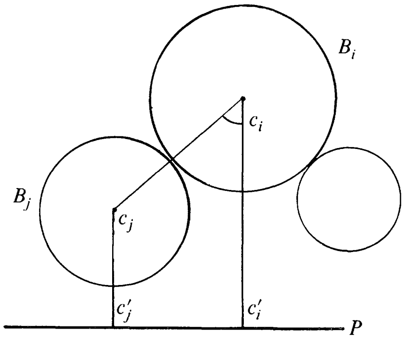


FIGURE 1. $\angle c_i' c_i c_j < \pi/2$.

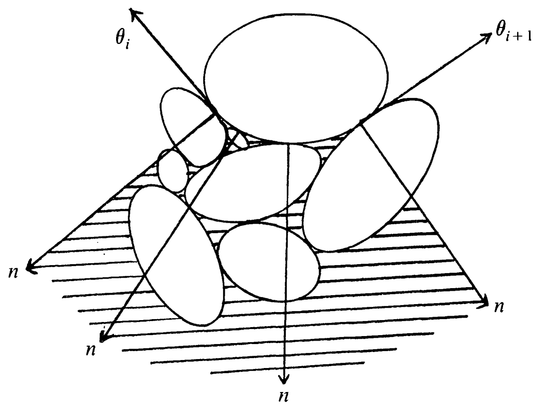


FIGURE 2. **The normal vectors lie within a closed semicircle.**

With this lemma, we may prove the theorem as follows. Let us suppose that the set of balls which are movable to infinity has fewer than $n+1$ elements. The centers of these balls are all contained in some hyperplane P, and there must exist some ball B_i whose center c_i is at a maximum distance (possibly zero) from P. Suppose that B_i touches some other ball B_j, and define c_i', c_j' to be the projections onto P of c_i and c_j; then c_i, c_i', c_j and c_j' form a trapezoid (possibly degenerate), with $|\vec{c}_i - \vec{c}_i'| \geqslant |\vec{c}_j - \vec{c}_j'|$ (FIGURE 1).

Thus, $\angle c_i' c_i c_j \leqslant \pi/2$; and as $(\vec{c}_j - \vec{c}_i)$ is parallel to n_{ij}, we see that $\vec{n}_{ij} \cdot (\vec{c}_i - \vec{c}_i') \leqslant 0$ for each B_j that touches B_i. By the Lemma, B_i is movable by translation parallel to the vector $(\vec{c}_i - \vec{c}_i')$. As this translation increases the distance from the center of B_i to its projection onto P, the conditions for movability clearly remain true, and B_i is movable to infinity.

But as P contains the center of every ball that is movable to infinity, $|\vec{c}_i - \vec{c}_i'| = 0$; and as the center of B_i is selected to be at a maximum distance from P, the center of every sphere in the collection lies in P. For any two adjacent balls B_i, B_j, the normal vector $\vec{n}_{ij}$ to the surface of B_i at their point of contact is parallel to $(\vec{c}_j - \vec{c}_i)$, and so lies in P; and, by the Lemma, every ball in the collection is movable by translation along a vector normal to P. Hence, the number of elements in the collection which are movable to infinity is greater than or equal to $\min(m, n+1)$.

The proof of the preceding theorem depends explicitly on the fact that the bodies are spherical, which allows us to associate the normal vectors with the vectors joining centers; where this is not the case, other methods must be used, as in the following theorem.

THEOREM 2. *In any finite collection of three or more smooth convex bodies in* $\mathbb{R}^2$, *intersecting at most in their boundaries, there exist at least three elements which are movable.*

Proof. Between each adjacent pair of bodies on the periphery of the collection there exists a common tangent line, at an angle θ_i (measured clockwise from an arbitrary ray). These angles are indexed cyclically in order of their position on the boundary: $\theta_0, \theta_1, \ldots, \theta_n$, with θ_0 and θ_n representing the same tangent line, but with the numerical value of θ_n greater by 2π as a result of the complete circuit about the perimeter of the collection that separates them.

Then,

$$\sum_{i=0}^{n-1} (\theta_{i+1} - \theta_i) = 2\pi.$$

As none of the differences $\theta_{i+1} - \theta_i$ can be greater than π, at least three of them must be nonnegative. We shall see that the bodies between these pairs of tangent lines must be movable. The angles of the normal vectors to such a body at any point of contact with any other body in the collection must lie between $\theta_{i+1} + (\pi/2)$ and $\theta_i - (\pi/2)$. But, if $(\theta_{i+1} - \theta_i) \geqslant 0$, these normal vectors lie within a closed semicircle (FIGURE 2), and, by the Lemma, the body is movable.

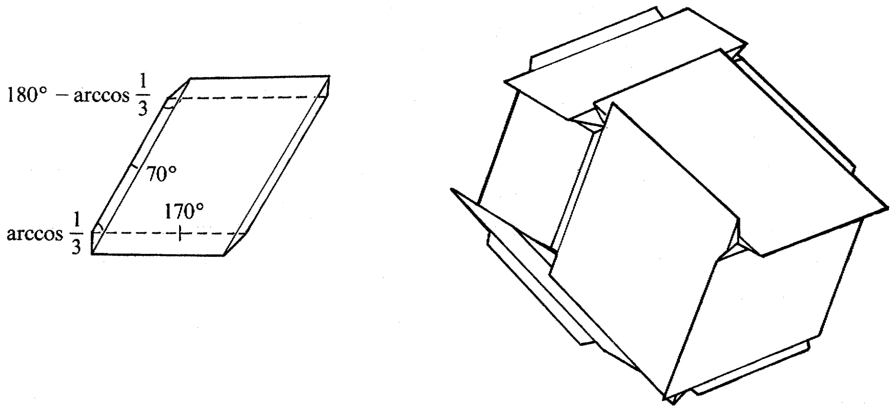


FIGURE 3. **Twelve convex tiles, none of which may be moved without disturbing the others.**

(This result can in fact be strengthened slightly, by omitting the word "smooth" from the statement of the Theorem; this will, however, not be shown here.)

Theorem 2 cannot be generalized to spaces of arbitrary dimension. It is not difficult to exhibit collections of convex bodies in $\mathbb{R}^3$ such that none can be moved without disturbing the others. For example, twelve thin "tiles" which correspond to the faces of a rhombic dodecahedron, with one pair of narrow faces sloping outward (making an angle of 170° with the rhombic face) and the other pair sloping inward (with a dihedral angle of 70°) may be so interlocked (see FIGURE 3).

It is not known how few convex bodies suffice for such a counterexample, nor what other restrictions on bodies in $\mathbb{R}^3$ or higher-dimensional spaces necessitate the existence of a removable object. It is tempting to conjecture that there might exist in $\mathbb{R}^n$ a set of convex, or at least star-shaped bodies, so interlocked that one cannot be removed even while disturbing the others—in other words, that there might exist an impossible "Chinese puzzle" with convex or star-shaped pieces. We see, however, that this is not so; while such a puzzle may not have pieces individually movable, the following proposition shows that it will always fall apart if shaken.

THEOREM 3. *Any collection of star-shaped bodies in $\mathbb{R}^n$ can be separated by simultaneous translations.*

Proof. Let each B_i be star-shaped about s_i. We show that if each B_i is translated by $t\vec{s}_i$, no intersection results for any $t \in [0, \infty)$.

Let $t = b/(1-b)$, $b \in [0,1)$. For $x_i \in B$, the translation $t\vec{s}_i$ maps x_i to $T(x_i) = b/(1-b)\vec{s}_i + \vec{x}_i$.

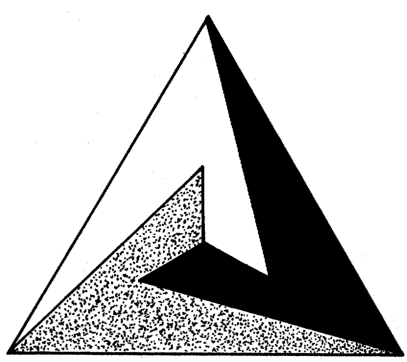


FIGURE 4. **Three star-shaped bodies, none of which may be moved without moving the others.**

Dilating the entire collection by a factor of $(1-b)$,

$$(1-b)T(\vec{x}_i) = b\vec{s}_i + (1-b)\vec{x}_i$$
$$= \vec{s}_i + (1-b)(\vec{x}_i - \vec{s}_i).$$

Thus, the translations are equivalent to a linear contraction of each body about a point about which it is star-shaped, followed by a global dilation. This cannot lead to intersection. (The writer is indebted to N. Higson, who suggested this method of proof.)

On the other hand, even in $\mathbb{R}^2$, it is possible for a collection of as few as three star-shaped bodies not to permit the removal of one body without disturbing the others (FIGURE 4).

The author is grateful to the referees for several improvements in the paper.

Reference

[1] W. Moser, Research Problems in Discrete Geometry, 1981, Dept. of Mathematics, McGill University, Montreal.

Curve Fitting of Averaged Data

VACLAV KONECNY
PETER R. GWILT
Ferris State College
Big Rapids, MI 49307

Two students independently conduct an experiment to determine the acceleration of a particle moving with uniform acceleration. They record the distance (y) travelled as a function of time (t). The data (t, y) obtained in their experiments are arranged in the following table.

Student 1	(1,2)	(2,6)	(3,8)
Student 2	(1,1)	(2,5)	(3,14)

The students know that $y = at^2/2$, where a is the acceleration. They assume that the initial velocity and distance equal zero. They set $\theta = a/2$. Thus $y = f(t,\theta) = \theta t^2$, and θ is to be determined.

Each student uses his own data to obtain a least squares estimate of θ. Drawing upon their previous knowledge of statistics they proceed to minimize S_i, the sum of the squares of differences (where i represents the student):

$$S_i = \sum_{j=1}^{N} \left[f(t_j,\theta) - y_{ji}\right]^2 \quad i = 1,2,\dots,n \tag{1}$$

where N is the number of time points when measurements are taken, t_j are the time points at which the measurements are taken, $f(t,\theta)$ is the function to be fitted with the parameter θ to be estimated, y_{ji} $(j = 1,2,\dots,N, i = 1,2,\dots,n)$ is the measured response of the ith student at time t_j, n is the number of experimenting students. Also, the **average measured response** at time t_j is defined as

$$\bar{y}_j = \frac{1}{n_j} \sum_{i(j)} y_{ji} \quad j = 1,2,\dots,N \tag{2}$$

where n_j is the number of measurements available from n students at a given time t_j ($n_j = n$ if n_j is constant; n_j may not be constant if the data from some students are missing), and $\Sigma_{i(j)}$ indicates that the sum in (2) is taken over all the n_j values y_{ji} supplied by students.

Dilating the entire collection by a factor of $(1-b)$,

$$(1-b)T(\vec{x}_i) = b\vec{s}_i + (1-b)\vec{x}_i$$
$$= \vec{s}_i + (1-b)(\vec{x}_i - \vec{s}_i).$$

Thus, the translations are equivalent to a linear contraction of each body about a point about which it is star-shaped, followed by a global dilation. This cannot lead to intersection. (The writer is indebted to N. Higson, who suggested this method of proof.)

On the other hand, even in $\mathbb{R}^2$, it is possible for a collection of as few as three star-shaped bodies not to permit the removal of one body without disturbing the others (FIGURE 4).

The author is grateful to the referees for several improvements in the paper.

Reference

[1] W. Moser, Research Problems in Discrete Geometry, 1981, Dept. of Mathematics, McGill University, Montreal.

Curve Fitting of Averaged Data

VACLAV KONECNY
PETER R. GWILT
Ferris State College
Big Rapids, MI 49307

Two students independently conduct an experiment to determine the acceleration of a particle moving with uniform acceleration. They record the distance (y) travelled as a function of time (t). The data (t, y) obtained in their experiments are arranged in the following table.

Student 1	(1,2)	(2,6)	(3,8)
Student 2	(1,1)	(2,5)	(3,14)

The students know that $y = at^2/2$, where a is the acceleration. They assume that the initial velocity and distance equal zero. They set $\theta = a/2$. Thus $y = f(t,\theta) = \theta t^2$, and θ is to be determined.

Each student uses his own data to obtain a least squares estimate of θ. Drawing upon their previous knowledge of statistics they proceed to minimize S_i, the sum of the squares of differences (where i represents the student):

$$S_i = \sum_{j=1}^{N} \left[f(t_j,\theta) - y_{ji}\right]^2 \quad i = 1,2,\dots,n \tag{1}$$

where N is the number of time points when measurements are taken, t_j are the time points at which the measurements are taken, $f(t,\theta)$ is the function to be fitted with the parameter θ to be estimated, y_{ji} $(j = 1,2,\dots,N, i = 1,2,\dots,n)$ is the measured response of the ith student at time t_j, n is the number of experimenting students. Also, the **average measured response** at time t_j is defined as

$$\bar{y}_j = \frac{1}{n_j} \sum_{i(j)} y_{ji} \quad j = 1,2,\dots,N \tag{2}$$

where n_j is the number of measurements available from n students at a given time t_j ($n_j = n$ if n_j is constant; n_j may not be constant if the data from some students are missing), and $\Sigma_{i(j)}$ indicates that the sum in (2) is taken over all the n_j values y_{ji} supplied by students.

The students obtain, by calculus, the following normal equation from (1):

$$\sum_{j=1}^{N}\left[f(t_j,\theta)-y_{ji}\right]\frac{\partial f}{\partial\theta}=0, \tag{3}$$

and using their function ($y=\theta t^2$) they get

$$\sum_{j=1}^{3}\left[\theta t_j^2-y_{ji}\right]t_j^2=0 \quad i=1,2. \tag{4}$$

Substituting into this equation the data actually obtained, student 1 gets

$$[\theta-2]+[4\theta-6]4+[9\theta-8]9=0$$

from which $\theta_1=1$. Student 2 gets

$$[\theta-1]+[4\theta-5]4+[9\theta-14]9=0$$

from which $\theta_2=3/2$.

The students get together and decide to pool their data in order to get a better estimate of the parameter. How should they proceed? They decide to use the following methods.

Method 1. Every response for every individual is used in the least squares procedure.
Method 2. At each time point the responses from all the individuals are averaged and the least squares procedure is applied to the average responses.
Method 3. The least squares fit is performed for each individual to obtain a parameter for each individual; these parameters are then averaged over all individuals.

For Method 1, the students minimize the sum

$$S=\sum_{j=1}^{N}\sum_{i(j)}\left[f(t_j,\theta)-y_{ji}\right]^2 \tag{5}$$

while for Method 2 they minimize the sum

$$S=\sum_{j=1}^{N}\left[f(t_j,\theta)-\bar{y}_j\right]^2. \tag{6}$$

For Method 3, they calculate

$$\theta=\frac{1}{n}\sum_{i=1}^{n}\theta_i$$

where θ_i is the parameter obtained by the ith student.

The normal equation corresponding to (5) is

$$\sum_{j=1}^{N}\left[n_j f(t_j,\theta)-n_j\bar{y}_j\right]\frac{\partial f}{\partial\theta}=0 \tag{7}$$

while the normal equation corresponding to (6) is

$$\sum_{j=1}^{N}\left[f(t_j,\theta)-\bar{y}_j\right]\frac{\partial f}{\partial\theta}=0. \tag{8}$$

Substituting the data into (7) the students get

$$[2\theta-3]+[2(4\theta)-11]4+[2(9\theta)-22]9=0$$

which gives $\theta=5/4$. The students immediately observe that (7) and (8) are equivalent in their experiments where $n_j=2$ in the normal equation (7). This parameter drops out giving an equation identical to equation (8). Thus Method 1 and Method 2 give the same solution. (This may not be surprising if we recall how the regression line was introduced, see Freedman [**1**].) But they also observe that even if n_j were not constant, Method 2 can be equivalent to Method 1 if one considers the weighted average with the weight equal to the number of measurements n_j at the given moment t_j. In this case, minimizing

$$\sum_{j=1}^{N} n_j\left[f(t_j,\theta)-\bar{y}_j\right]^2$$

yields (7). This is also true for

$$\sum_{j=1}^{N}\sum_{i(j)}\left(f(t_j,\theta)-y_{ji}\right)^2=\sum_{j=1}^{N} n_j\left(f(t_j,\theta)-\bar{y}_j\right)^2+\sum_{j=1}^{N}\sum_{i(j)}\left(y_{ji}-\bar{y}_j\right)^2.$$

To complete Method 3, $\theta=(\theta_1+\theta_2)/2=(1+3/2)/2=5/4$. Thus all three methods give the same answer.

The students wonder what happens if n_j is not constant. For clarity, they simplify the previous data set as follows. Student 1: (2,6), (3,8) and student 2: (1,1), (2,5). They already know that Method 1 and Method 2 yield the same answer, provided that the weighted least squares technique is applied. They use the same function as before and employ equation (4) to get $\theta_1=96/97$ and $\theta_2=21/17$. Method 1 and Method 2 now both yield $\theta=39/38$. But using Method 3 they get $\theta=(\theta_1+\theta_2)/2=3669/3298$, so in this case, Method 3 is not equivalent to Method 1 or 2. Weighted least squares, using weight equal to n_j, would not help because they had the same number of measurements, namely, two. They would have to introduce special weights depending on the function.

Those students with a more extensive statistical background might also be curious about the equivalence of the three methods if $f(t,\theta)$ is not linear in θ. Therefore, they make up the data as follows:

Student 1	(1,0)	(2,1)	(3,2)
Student 2	(1,1)	(2,1)	(3,3)

and now use a function $f(t,\theta)=\theta^2 t$. Again, use of equation (3) yields $\theta_1=(4/7)^{1/2}$ and $\theta_2=(6/7)^{1/2}$. Method 1 and Method 2 will again be equivalent for the reasons given above. Method 1 yields $\theta=(5/7)^{1/2}$ and Method 3 yields $(2+6^{1/2})/2(7)^{1/2}$. Values obtained by Method 3 again appear to differ from those values obtained by Methods 1 and 2.

If we summarize the observations it appears that all three methods are equivalent if $n_j=n$ and f is a linear function in θ. The equivalence of Method 2 and Method 3 was indeed proved by Jacquez [**2**] for a function f linear in a parameter θ.

We shall generalize the results to more than one-parameter functions. Before we do that let us study another experiment of the students. This time, they conduct an experiment to determine the initial velocity and the acceleration of a particle moving with uniform acceleration (experimental conditions are the same for both students). Again they record the distance (y) traveled as a function of time (t). The data (t,y) obtained in their experiments are arranged in the following table (data were made up).

Student 1	(1,4)	(2,9)	(3,22)
Student 2	(1,3)	(2,11)	(3,20)

The motion of a particle is described by the function $y=\theta_1 t+\theta_2 t^2$, where θ_1 is the initial velocity and $2\theta_2$ is the acceleration.

Each student uses his own data to obtain a least squares estimate of θ_1 and θ_2. Minimizing

$$\sum_{j=1}^{N}\left[f(t_j,\theta_1,\theta_2)-y_{ji}\right]^2$$

they obtain the following normal equations:

$$\sum_{j=1}^{N}\left[f(t_j,\theta_1,\theta_2)-y_{ji}\right]\frac{\partial f}{\partial\theta_k}=0 \qquad k=1,2$$

or

$$\sum_{j=1}^{3}\left[\theta_1 t_j+\theta_2 t_j^2-y_{ji}\right]t_j=0$$

$$\sum_{j=1}^{3}\left[\theta_1 t_j+\theta_2 t_j^2-y_{ji}\right]t_j^2=0.$$

Substituting the data into these two equations student 1 gets

$$[\theta_1+\theta_2-4]+[2\theta_1+4\theta_2-9]2+[3\theta_1+9\theta_2-22]3=0$$

$$[\theta_1+\theta_2-4]+[2\theta_1+4\theta_2-9]4+[3\theta_1+9\theta_2-22]9=0$$

or

$$14\theta_1+36\theta_2-88=0$$

$$36\theta_1+98\theta_2-238=0$$

from which $\theta_{11}=56/76$ and $\theta_{12}=164/76$ where the first subscript represents the student and the second subscript represents a parameter. In a similar way, student 2 gets $\theta_{21}=158/76$ and $\theta_{22}=118/76$. Method 1 and Method 2 both give the following equations

$$28\theta_1+72\theta_2-173=0$$

$$72\theta_1+196\theta_2-465=0$$

from which $\theta_1=107/76$ and $\theta_2=141/76$. Method 3 yields

$$\theta_1=(\theta_{11}+\theta_{21})/2=(56+158)/(76\times 2)=107/76$$

$$\theta_2=(\theta_{12}+\theta_{22})/2=(164+118)/(76\times 2)=141/76.$$

Thus again all three methods give the same answer; this is in agreement with our earlier observations for one-parameter functions. We now establish our observations as a mathematical fact.

We consider $f(t,\underline{\theta})=f(t,\theta_1,\theta_2,\dots,\theta_M)$ the function to be fitted with parameters $\underline{\theta}$ which are sought. The number of parameters, M, is less than or equal to N. Method 1 yields the equation

$$\sum_{j=1}^{N}\left[n_j f(t_j,\underline{\theta})-n_j\bar{y}_j\right]\frac{\partial f}{\partial\theta_k}=0\quad k=1,2,\dots,M \tag{9}$$

while Method 2 gives the equation

$$\sum_{j=1}^{N}\left[f(t_j,\underline{\theta})-\bar{y}_j\right]\frac{\partial f}{\partial\theta_k}=0\quad k=1,2,\dots,M. \tag{10}$$

Method 1 and Method 2 are equivalent if $n_j=n$ or $f(t_j,\underline{\theta})=\bar{y}_j$ for every j; i.e., a perfect fit of the average data is obtained. If we consider weighted least squares in Method 2 with the weight equal to n_j we obtain, as in the case of one parameter function, that Method 1 is equivalent to Method 2.

To compare Method 3 with Method 1, assume $n_j=n$. For Method 3, let $\underline{\theta}_i$ be the set of M parameters for the ith subject obtained by the least squares fit to $f(t,\underline{\theta}_i)$. Thus

$$S=\sum_{j=1}^{N}\left[f(t_j,\underline{\theta}_i)-y_{ji}\right]^2,$$

and for the minimum of S we get

$$\sum_{j=1}^{N}\left[f(t_j,\underline{\theta}_i)-y_{ji}\right]\frac{\partial f}{\partial\theta_k}=0,\quad k=1,2,\dots,M. \tag{11}$$

Notice that $\partial f/\partial\theta_k$ at t_j is independent of i since the same function is used to describe all the individuals. Expand $f(t_j,\underline{\theta}_i)$ into a Taylor series in $\underline{\theta}_i$ about $\underline{\theta}_0$ where $\underline{\theta}_0$ is some vector:

$$f(t_j,\underline{\theta}_i)=f(t_j,\underline{\theta}_0)+\sum_{m=1}^{M}\left(\frac{\partial f}{\partial\theta_m}\right)_{\underline{\theta}_0}(\theta_{im}-\theta_{0m})+R_i, \tag{12}$$

where R_i is the remainder. In equation (12), θ_{im} is the mth parameter of the ith subject and θ_{0m} is the mth parameter from $\underline{\theta}_0$. When $f(t_j,\underline{\theta}_i)$ is linear in $\underline{\theta}_i$, then $R_i=0$. Also, note in (12) that $(\partial f/\partial\theta_m)_{\underline{\theta}_0}$ is independent of i. Substituting (12) into (11) we get

$$\sum_{j=1}^{N}\left[f(t_j,\underline{\theta}_0)+\sum_{m=1}^{M}\left(\frac{\partial f}{\partial\theta_m}\right)_{\underline{\theta}_0}(\theta_{im}-\theta_{0m})+R_i-y_{ji}\right]\frac{\partial f}{\partial\theta_k}=0. \tag{13}$$

It is convenient to choose

$$\theta_{0m}=\frac{1}{n}\sum_{i=1}^{n}\theta_{im}.$$

Add equations (13) over i and change the order of summation to get

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\left[f(t_j,\underline{\theta}_0)-y_{ji}\right]\frac{\partial f}{\partial\theta_k}+\sum_{j=1}^{N}\sum_{m=1}^{M}\left(\frac{\partial f}{\partial\theta_m}\right)_{\underline{\theta}_0}\sum_{i=1}^{n}(\theta_{im}-\theta_{0m})\frac{\partial f}{\partial\theta_k}=0.$$

The second term in this equation is equal to 0 and thus

$$\sum_{j=1}^{N}\left[nf(t_j,\underline{\theta}_0)-\sum_{i=1}^{n}y_{ji}\right]\frac{\partial f}{\partial\theta_k}=0. \tag{14}$$

If we compare (14) and (9) we see that they are equivalent for $n_j=n$. Thus we conclude that Method 3 is equivalent to Methods 1 and 2 if $f(t,\underline{\theta})$ is a linear function in $\underline{\theta}$ and $n_j=n$. Method 2 is equivalent to Method 1 if the number of data n_j at each t_j is equal to n or if the graph of $f(t,\underline{\theta})$ goes through the average values of each group of data.

It is uncertain which of the three methods best estimates the population mean parameter values. However it is sufficient to offer this *caveat*: each method may give very different estimates of the mean parameter values if the conditions described above are not satisfied.

References

[1] D. Freedman, Statistics, W. W. Norton, 1978, pp. 146–167.

[2] J. A. Jacquez, Compartmental Analysis in Biology and Medicine, Elsevier, 1972, pp. 119–120.

Reliability and Runs of Ones

Richard C. Bollinger

Behrend College, Pennsylvania State University

Erie, PA 16563

The intensive coverage of NASA's space activities makes us all aware of the critical need for highly reliable components and systems. We've heard many times how the effects of this or that malfunction were overcome by finding an equivalent parallel path for a signal, or perhaps even having a complete duplicate system (or two) as a backup. We describe here a method for increasing the reliability of systems of n independent "components" (which may themselves be subsystems) and discuss some of the probabilistic and combinatorial aspects of the method.

Notice that $\partial f/\partial\theta_k$ at t_j is independent of i since the same function is used to describe all the individuals. Expand $f(t_j,\underline{\theta}_i)$ into a Taylor series in $\underline{\theta}_i$ about $\underline{\theta}_0$ where $\underline{\theta}_0$ is some vector:

$$f(t_j,\underline{\theta}_i)=f(t_j,\underline{\theta}_0)+\sum_{m=1}^{M}\left(\frac{\partial f}{\partial\theta_m}\right)_{\underline{\theta}_0}(\theta_{im}-\theta_{0m})+R_i, \tag{12}$$

where R_i is the remainder. In equation (12), θ_{im} is the mth parameter of the ith subject and θ_{0m} is the mth parameter from $\underline{\theta}_0$. When $f(t_j,\underline{\theta}_i)$ is linear in $\underline{\theta}_i$, then $R_i=0$. Also, note in (12) that $(\partial f/\partial\theta_m)_{\underline{\theta}_0}$ is independent of i. Substituting (12) into (11) we get

$$\sum_{j=1}^{N}\left[f(t_j,\underline{\theta}_0)+\sum_{m=1}^{M}\left(\frac{\partial f}{\partial\theta_m}\right)_{\underline{\theta}_0}(\theta_{im}-\theta_{0m})+R_i-y_{ji}\right]\frac{\partial f}{\partial\theta_k}=0. \tag{13}$$

It is convenient to choose

$$\theta_{0m}=\frac{1}{n}\sum_{i=1}^{n}\theta_{im}.$$

Add equations (13) over i and change the order of summation to get

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\left[f(t_j,\underline{\theta}_0)-y_{ji}\right]\frac{\partial f}{\partial\theta_k}+\sum_{j=1}^{N}\sum_{m=1}^{M}\left(\frac{\partial f}{\partial\theta_m}\right)_{\underline{\theta}_0}\sum_{i=1}^{n}(\theta_{im}-\theta_{0m})\frac{\partial f}{\partial\theta_k}=0.$$

The second term in this equation is equal to 0 and thus

$$\sum_{j=1}^{N}\left[nf(t_j,\underline{\theta}_0)-\sum_{i=1}^{n}y_{ji}\right]\frac{\partial f}{\partial\theta_k}=0. \tag{14}$$

If we compare (14) and (9) we see that they are equivalent for $n_j=n$. Thus we conclude that Method 3 is equivalent to Methods 1 and 2 if $f(t,\underline{\theta})$ is a linear function in $\underline{\theta}$ and $n_j=n$. Method 2 is equivalent to Method 1 if the number of data n_j at each t_j is equal to n or if the graph of $f(t,\underline{\theta})$ goes through the average values of each group of data.

It is uncertain which of the three methods best estimates the population mean parameter values. However it is sufficient to offer this *caveat*: each method may give very different estimates of the mean parameter values if the conditions described above are not satisfied.

References

[1] D. Freedman, Statistics, W. W. Norton, 1978, pp. 146–167.

[2] J. A. Jacquez, Compartmental Analysis in Biology and Medicine, Elsevier, 1972, pp. 119–120.

Reliability and Runs of Ones

Richard C. Bollinger

Behrend College, Pennsylvania State University

Erie, PA 16563

The intensive coverage of NASA's space activities makes us all aware of the critical need for highly reliable components and systems. We've heard many times how the effects of this or that malfunction were overcome by finding an equivalent parallel path for a signal, or perhaps even having a complete duplicate system (or two) as a backup. We describe here a method for increasing the reliability of systems of n independent "components" (which may themselves be subsystems) and discuss some of the probabilistic and combinatorial aspects of the method.

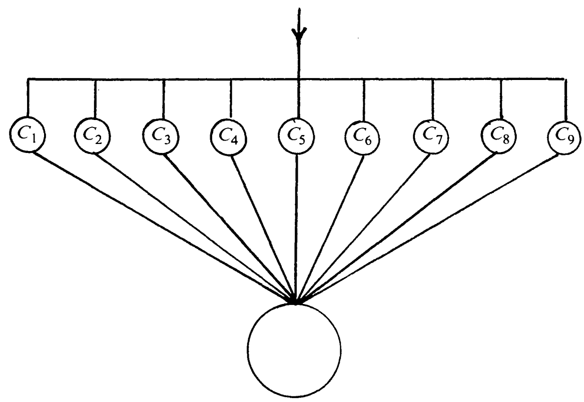


FIGURE 1(a). **The original 9-component system.**

To make things more concrete, consider a system of 9 components, each of which has probability p of operating to "process" some input and transmit the result to a central location, as in FIGURE 1(a). The system is assumed to fail if any component fails to operate. The failure probability of this system is $1-p^9$, and if p is close enough to 1, failure may be very unlikely. But the whole system fails if just one component fails. What if it's very expensive or difficult to have p large enough to make the system as reliable as we require?

Reliability may be increased without duplicating the system by using what reliability engineers call a **consecutive-k-out-of-n: F system**, which is a system of n linearly ordered, independent components, each of which operates with probability p, and such that the system fails when and only when at least k consecutive components fail. We denote the **failure probability** of the system by $P_F(n,k)$; the **reliability** of the system is $1-P_F(n,k)$.

Suppose the original system of our example is replaced by a consecutive 3-out-of-10: F system, as shown in FIGURE 1(b). The circles in the upper row represent (perhaps simpler) elements E_1-E_{10} which "protect" components C_1-C_9 in such a way that C_1-C_9 are no longer subject to failure. The elements E_1-E_{10} themselves, however, now each operate with probability p. Inspection shows that the system now fails when and only when three or more consecutive E's fail.

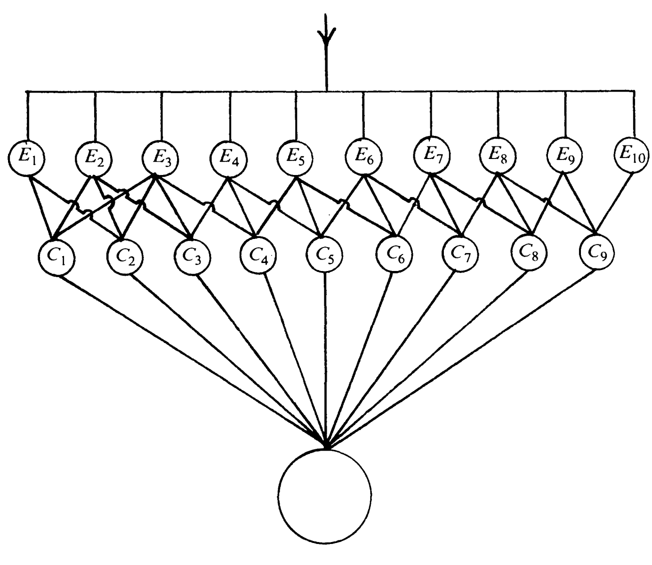


FIGURE 1(b). **The modified system: consecutive-3-out-of-10: *F*.**

The failure probability of such a system may be calculated by converting the problem to a combinatorial one involving binary numbers of length n. In a binary number, we let 1 represent a failure; then we need to count the binary numbers of length n (including initial 0's) which have a string of k or more *consecutive* 1's. We denote by $r_j^{(k)}$ the number of binary numbers of length n containing the digit 1 a total of j times, with at least k of these 1's consecutive. Since the probability of a 1 in each position is q, the probability of j of these occurring is $p^{n-j}q^j$ (where $q = 1 - p$), so the failure probability is

$$P_F(n, k) = \sum_{j=k}^{n} r_j^{(k)} p^{n-j} q^j. \tag{1}$$

Now we consider how to obtain the coefficients $r_j^{(k)}$. Although generating functions can be used, the numbers $r_j^{(k)}$ can be computed in a surprisingly simple and computationally effective way.

For any $k \geqslant 2$, let T_k be the **generalized Pascal triangle** constructed as follows:

(1) the rows are indexed by $n \geqslant 0$ and the columns by $j \geqslant 0$;
(2) the $n = 0$ row has a 1 followed by zeros, and the $n = 1$ row has k 1's followed by zeros;
(3) each entry in the succeeding rows is the sum of the entry just above it and that entry's $(k - 1)$ immediate left neighbors.

If we denote the n, j entry of T_k by $C_k(n, j)$, then property (3) says that

$$C_k(n, j) = \sum_{i=j-k+1}^{j} C_k(n-1, i), \text{ where } C_k(n-1, i) = 0 \text{ for } i < 0.$$

Note that T_2 is Pascal's triangle, so that $C_2(n, j) = \binom{n}{j}$. FIGURE 2 shows a portion of T_3, which we'll use in our example, after proving two simple results.

THEOREM 1. *Let $g_k(n, j)$ denote the number of binary numbers of length n which contain a total j of 1's but do **not** have k consecutive 1's. Then*

$$g_k(n, j) = C_k(n - j + 1, j).$$

Proof. For $0 \leqslant j \leqslant k - 1$, it is clear from the definitions that

$$g_k(n, j) = \binom{n}{j} = C_k(n - j + 1, j),$$

and for $n \geqslant k$, $g_k(n, n) = C_k(1, n) = 0$. Now suppose $k \leqslant j < n$, and suppose that the result is correct for all $g_k(m, j)$ with $m < n$. The numbers of the indicated type which end in 0 are enumerated by

$$g_k(n-1, j) = C_k(n - j, j);$$

those which end in 01 are enumerated by

$$g_k(n-2, j-1) = C_k(n - j, j - 1);$$

those which end in 011 are enumerated by

$$g_k(n-3, j-2) = C_k(n - j, j - 2);$$

and so on. Thus the indicated numbers are enumerated (by their terminal strings of 1's) by

$$g_k(n, j) = C_k(n - j, j) + C_k(n - j, j - 1) + \cdots + C_k(n - j, j - k + 1).$$

Since this is exactly the relation that defines the elements of T_k, we have

$$g_k(n, j) = C_k(n - j + 1, j).$$

THEOREM 2. *The coefficients $r_j^{(k)}$ in (1) are given by*

$$r_j^{(k)} = \binom{n}{j} - C_k(n - j + 1, j). \tag{2}$$

$n \backslash j$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0
2	1	2	3	2	1	0	0	0	0	0
3	1	3	6	7	6	3	1	0	0	0
4	1	4	10	16	19	16	10	4	1	0
5	1	5	15	30	45	51	45	30	15	5
6	1	6	21	50	90	126	141	126	90	50
7	1	7	28	77	161	266	357	393	357	266
8	1	8	36	112	266	504	784	1016	1107	1016
9	1	9	45	156	414	882	1554	2304	2907	3139
10	1	10	55	210	615	1452	2850	4740	6765	8350

FIGURE 2. **The generalized Pascal Triangle for $k = 3$ whose n, j entry is $C_3(n, j)$.**

Proof. Since $\binom{n}{j}$ is the total number of binary numbers of length n which have j 1's, the result follows immediately from Theorem 1.

We now have formula (1) for $P_F(n, k)$, with coefficients given by (2). Let's go back to our example and compare the straight in-line design with the modified consecutive-3-out-of-10: F system. We have

$$r_j^{(3)} = \binom{10}{j} - C_3(11-j, j),\ 3 \leqslant j \leqslant 10,$$

and (using FIGURE 2)

$$P_F(10,3) = 8p^7q^3 + 49p^6q^4 + 126p^5q^5 + 165p^4q^6 + 116p^3q^7 + 45p^2q^8 + 10pq^9 + q^{10}.$$

To exaggerate the comparison a little, let's take $p = .5$. The in-line system has a reliability of $(.5)^9 \doteq .002$, while the reliability of the 3-of-10 system is $1 - .508 = .492$, not bad for such shaky components. More realistically, for $p = .95$ the reliability of the in-line system is still only .63, while that of the 3-of-10 system is better than .99, a difference that would often justify design modification.

A Bijection in the Theory of Derangements

HERBERT S. WILF
University of Pennsylvania
Philadelphia, PA 19104

The following is a well-known fact about permutations.

THEOREM. *If n is odd, then there is one more permutation of n letters with exactly* 1 *fixed point than with no fixed points*; *"more" is replaced by "less" if n is even.*

For example, there are 45 permutations of the set $\{1,2,3,4,5\}$ having exactly one fixed point, and 44 permutations of that set having no fixed points.

The theorem is usually proved by direct evaluation of the two quantities, using the sieve method (see, e.g., [**2**], chap. 4). In this note we will first give a bijective proof of the theorem, and then show how to carry out the mapping and its inverse by an interesting pair of co-recursive routines. Another constructive proof has been given by Remmel [**1**]. Ours is somewhat simpler, but does not generalize, as his does, to the q-analogues that he studied.

n \ j	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0
2	1	2	3	2	1	0	0	0	0	0
3	1	3	6	7	6	3	1	0	0	0
4	1	4	10	16	19	16	10	4	1	0
5	1	5	15	30	45	51	45	30	15	5
6	1	6	21	50	90	126	141	126	90	50
7	1	7	28	77	161	266	357	393	357	266
8	1	8	36	112	266	504	784	1016	1107	1016
9	1	9	45	156	414	882	1554	2304	2907	3139
10	1	10	55	210	615	1452	2850	4740	6765	8350

FIGURE 2. **The generalized Pascal Triangle for $k = 3$ whose n, j entry is $C_3(n, j)$.**

Proof. Since $\binom{n}{j}$ is the total number of binary numbers of length n which have j 1's, the result follows immediately from Theorem 1.

We now have formula (1) for $P_F(n, k)$, with coefficients given by (2). Let's go back to our example and compare the straight in-line design with the modified consecutive-3-out-of-10: F system. We have

$$r_j^{(3)} = \binom{10}{j} - C_3(11-j, j), 3 \leqslant j \leqslant 10,$$

and (using FIGURE 2)

$$P_F(10,3) = 8p^7q^3 + 49p^6q^4 + 126p^5q^5 + 165p^4q^6 + 116p^3q^7 + 45p^2q^8 + 10pq^9 + q^{10}.$$

To exaggerate the comparison a little, let's take $p = .5$. The in-line system has a reliability of $(.5)^9 \doteq .002$, while the reliability of the 3-of-10 system is $1 - .508 = .492$, not bad for such shaky components. More realistically, for $p = .95$ the reliability of the in-line system is still only .63, while that of the 3-of-10 system is better than .99, a difference that would often justify design modification.

A Bijection in the Theory of Derangements

HERBERT S. WILF
University of Pennsylvania
Philadelphia, PA 19104

The following is a well-known fact about permutations.

THEOREM. *If n is odd, then there is one more permutation of n letters with exactly* 1 *fixed point than with no fixed points; "more" is replaced by "less" if n is even.*

For example, there are 45 permutations of the set $\{1,2,3,4,5\}$ having exactly one fixed point, and 44 permutations of that set having no fixed points.

The theorem is usually proved by direct evaluation of the two quantities, using the sieve method (see, e.g., [**2**], chap. 4). In this note we will first give a bijective proof of the theorem, and then show how to carry out the mapping and its inverse by an interesting pair of co-recursive routines. Another constructive proof has been given by Remmel [**1**]. Ours is somewhat simpler, but does not generalize, as his does, to the q-analogues that he studied.

In our definitions below, $[n]$ denotes the set $\{1,2,\ldots,n\}$, and S_n denotes the set of all n-permutations, i.e., the permutations of $[n]$. For $j=0,1,\ldots,n$, let $F_j^{(n)}$ be the set of n-permutations having exactly j fixed points, and let $f_j^{(n)}$ be its cardinality. Using this notation, our theorem can be stated succinctly:

THEOREM. *For all* $n, f_1^{(n)} = f_0^{(n)} - (-1)^n$.

To prove the theorem, we want a mapping which, if n is odd, associates with all but one permutation of n letters with one fixed point a permutation with no fixed points (and the opposite if n is even).

Let $G^{(n)}$ be the set of n-permutations with no fixed points in which the letter n belongs to a 2-cycle, and let g_n be its cardinality. The following three mappings of permutations will be used.

Cut: $G^{(n)} \to F_1^{(n-1)}$.
Let $\sigma \in G^{(n)}$, $t = \sigma(n)$. Then

$$(\mathbf{Cut} \circ \sigma)(j) = \begin{cases} \sigma(j) & \text{if } j \neq t,\ j \in [n-1] \\ t & \text{if } j = t. \end{cases}$$

Splice: $F_1^{(n-1)} \to G^{(n)}$.
Let t be the fixed point of σ. Then

$$(\mathbf{Splice} \circ \sigma)(j) = \begin{cases} n & \text{if } j = t \\ t & \text{if } j = n \\ \sigma(j) & \text{otherwise.} \end{cases}$$

Insert: $\{\sigma \in F_1^{(n)} | \sigma(n) \neq n\} \to F_0^{(n)}$.
Let $t \neq n$ be the fixed point of σ. Then

$$(\mathbf{Insert} \circ \sigma)(j) = \begin{cases} n & \text{if } j = t \\ t & \text{if } j = \sigma^{-1}(n) \\ \sigma(j) & \text{otherwise.} \end{cases}$$

To prove the theorem, we first note that **Splice** = **Cut**$^{-1}$, so it follows that **Cut** is a bijection. Thus

$$f_1^{(n-1)} = g_n. \tag{1}$$

We want to "match" subsets of $F_0^{(n)}$ and $F_1^{(n)}$, so next consider **Insert**, which is 1-1 from a subset of $F_1^{(n)}$ to a subset of $F_0^{(n)}$. The "unmatched" subset of $F_0^{(n)}$ (its elements are not images under **Insert**) is $G^{(n)}$, and by (1), its cardinality is $f_1^{(n-1)}$. The "unmatched" subset of $F_1^{(n)}$ (not in the domain of **Insert**) is the set of n-permutations whose unique fixed point is n, and clearly it has cardinality $f_0^{(n-1)}$.

Hence the excess $|F_0^{(n)}| - |F_1^{(n)}| = f_0^{(n)} - f_1^{(n)}$ must also be equal to $f_1^{(n-1)} - f_0^{(n-1)}$. Thus

$$f_0^{(n)} - f_1^{(n)} = -\left(f_0^{(n-1)} - f_1^{(n-1)}\right)$$

and the theorem follows since $f_0^{(1)} - f_1^{(1)} = -1$.

Now we consider the implementation of the bijection that proved the theorem. The proof implicitly defines two recursive functions, **Map** and **Mapinverse**.

The function **Map** accepts as input an n-permutation σ with exactly 1 fixed point, except that, if n is odd, the permutation

$$\sigma_n^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n-2 & n-1 & n \\ 2 & 1 & 4 & 3 & 6 & 5 & \ldots & n-1 & n-2 & n \end{pmatrix}$$

may not be input. Its output is an n-permutation **Map**(σ) with no fixed points.

The function **Mapinverse** accepts an n-permutation σ without fixed points, except that if n is even the permutation

$$\sigma_n^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & n-2 & n-1 & n \\ 2 & 1 & 4 & 3 & 6 & 5 & \dots & n-3 & n & n-1 \end{pmatrix}$$

is forbidden input. The output permutation **Mapinverse**(σ) has exactly 1 fixed point.

The two functions call each other recursively, and can be explicitly defined by means of the following three mappings.

Delete: $F_0^{(n)} - G^{(n)} \to F_1^{(n)}$.
Let $t = \sigma^{-1}(n)$, $u = \sigma^{-1}(t)$. Then

$$(\mathbf{Delete} \circ \sigma)(j) = \begin{cases} t & \text{if } j = t \\ n & \text{if } j = u \\ \sigma(j) & \text{otherwise.} \end{cases}$$

Grow: $S_{n-1} \to S_n$.

$$(\mathbf{Grow} \circ \sigma)(j) = \begin{cases} n & \text{if } j = n \\ \sigma(j) & \text{otherwise.} \end{cases}$$

Shrink: $\{\sigma \in S_n | \sigma(n) = n\} \to S_{n-1}$.

$$(\mathbf{Shrink} \circ \sigma)(j) = \sigma(j) \text{ for all } j \in [n-1].$$

The functions **Map** and **Mapinverse** are defined by the following co-recursive algorithms.

Function **Map**(σ)

Begin
  If $n = 1$ then return "forbidden input" else
    if n is not a fixed point of σ then return **Insert**(σ)
      else return **Splice(Mapinverse(Shrink(**σ**)))**
End.

Function **Mapinverse**(σ)

Begin
  If n is in a cycle of length > 2 then return **Delete**(σ)
    else return **Grow(Map(Cut(**σ**)))**
End.

The above program can easily be written in a recursive computer language. Interestingly, it is also quite easy to program in Basic, despite the mutual recursiveness. One keeps two linear arrays, representing the permutation and its inverse, and all functions act on these same two arrays.

The correspondence $F_0^{(n)} \leftrightarrow F_1^{(n)}$ is shown below for the case $n = 4$, where a permutation $\sigma \in S_4$ is represented by the sequence $\sigma(1)\sigma(2)\sigma(3)\sigma(4)$:

$F_0^{(4)}$		$F_1^{(4)}$
2143	$\leftrightarrow$	——
2341	$\leftrightarrow$	2431
2413	$\leftrightarrow$	4213
3142	$\leftrightarrow$	4132
3412	$\leftrightarrow$	2314
3421	$\leftrightarrow$	3241
4321	$\leftrightarrow$	3124
4312	$\leftrightarrow$	1342
4123	$\leftrightarrow$	1423

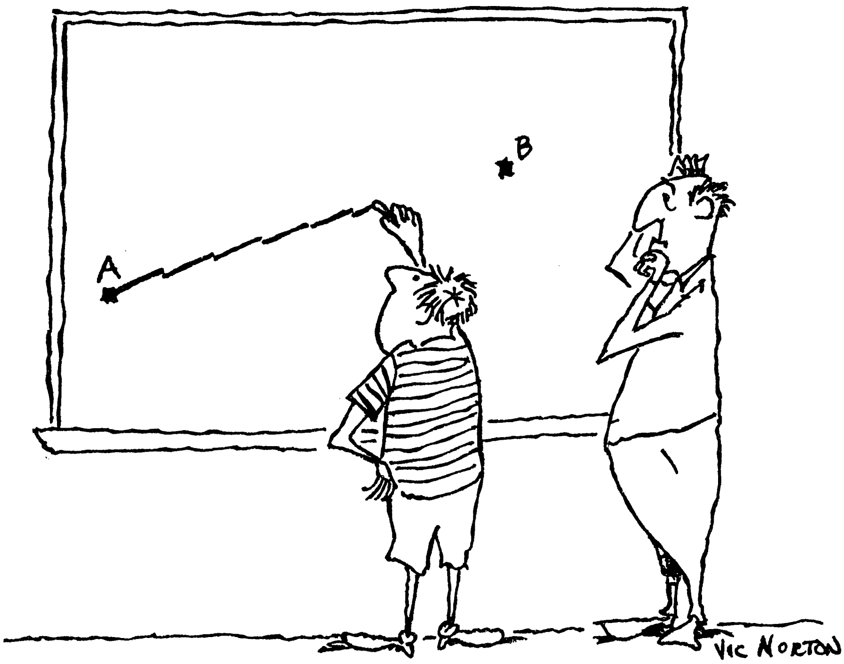
Research supported by the National Science Foundation.

I am pleased to acknowledge a number of comments from Dr. Donald Knuth that have greatly improved the clarity of the presentation.

References

[1] J. Remmel, On a recursion for the number of derangements, preprint.
[2] C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, 1968.

Cartoon without caption: The computer-age generation



GERALD PORTER
University of PA

VIC NORTON
Miami University

Readers are invited to submit appropriate captions to the editor. The five best (in the editor's opinion) will be published in our next issue.

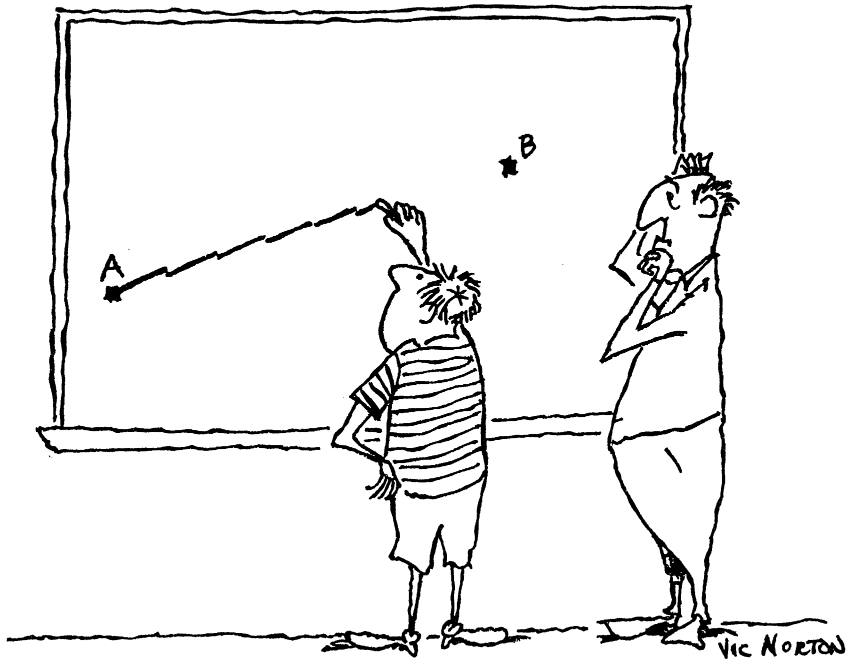
Research supported by the National Science Foundation.

I am pleased to acknowledge a number of comments from Dr. Donald Knuth that have greatly improved the clarity of the presentation.

References

[1] J. Remmel, On a recursion for the number of derangements, preprint.
[2] C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, 1968.

Cartoon without caption: The computer-age generation



GERALD PORTER
University of PA

VIC NORTON
Miami University

Readers are invited to submit appropriate captions to the editor. The five best (in the editor's opinion) will be published in our next issue.

PROBLEMS

LEROY F. MEYERS, Editor
G. A. EDGAR, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be submitted by June 1, 1984.

1182. Prove that

$$x \cot\frac{x}{2} - x\tan^3\frac{x}{2} < 2 \quad \text{for } 0 < x < \frac{\pi}{2}.$$

[*R. S. Luthar, University of Wisconsin Center, Janesville.*]

1183. *Solitario de la suerte* (chance solitaire) is a Spanish game similar to "clock." (Cf. in this MAGAZINE, problem 1066, solved v. 53 (1980) 184–185, and the articles by Jenkins & Miller, v. 54 (1981) 202–208, and by Ecker, v. 55 (1982) 42–43.) A version using American cards can be described mathematically as follows.

Forty-eight cards from a well-shuffled deck are placed face down in the first twelve columns of a 4×13 rectangular array. The rows are named after the suits (clubs, diamonds, spades, hearts) and the columns are named after the ranks (ace, two,...,king). The remaining four cards of the deck form the *hand*. A card is chosen from the hand and placed face up in the array in its proper position according to its suit and rank, replacing the face-down card, which is now placed in the array according to its suit and rank, etc., until a king is placed. The process is repeated for the second, third, and fourth cards in the hand. When all cards have been placed, the remaining face-down cards are turned over. The game is won if all cards are in their proper places, and lost otherwise.

(a) What is the probability of winning the game?

*(b) If the hand is known to contain exactly i kings, where $0 \leqslant i \leqslant 4$, what is the probability of winning? [*Julio Castiñeira, Segovia, Spain.*]

ASSISTANT EDITORS: DANIEL B. SHAPIRO and WILLIAM A. MCWORTER, JR., *The Ohio State University.*

*We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) will be placed next to a problem number to indicate that the proposer did not supply a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address. It is not necessary to submit duplicate copies.

Send all communications to the problems department to Leroy F. Meyers, Mathematics Department, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

1184. (a) Show, using results from elementary number theory only (such as congruence conditions), that the equations

$$x^2+5=y^3,\quad x^2+9=y^3,\quad x^2-71=y^7,\quad x^2+37=y^{11},\quad \text{and}\quad x^4+282=y^3$$

have no solutions in integers x, y.

(b) Of what general results are the examples in (a) corollaries? [*Barry Powell, Kirkland, Washington.*]

***1185.** Set $a_0=1$ and for $n\geqslant 1$, $a_n=a_{n'}+a_{n''}+a_{n'''}$, where $n'=[n/2]$, $n''=[n/3]$, and $n'''=[n/6]$. Find $\lim_{n\to\infty}a_n/n$. (Compare solution 1158, this issue, p. 49) [*Daniel A. Rawsthorne, Wheaton, Maryland.*]

Solutions

A Three-parameter Recurrence **May 1982**

1144. Define the numbers $A^n_{r,s}$ by $A^n_{0,0}=1$, $A^n_{r,-1}=A^n_{-1,s}=A^{-1}_{r,s}=0$, and

$$A^n_{r,s}=A^n_{r-1,s}+(n+r-s+1)A^n_{r,s-1}\text{ for } r+s>0 \text{ and } n\geqslant 0.$$

(a) Show that $A^n_{r,s}=A^{n+1}_{r-1,s}+nA^{n-1}_{r,s-1}$ for $r+s>0$ and $n\geqslant 0$.

(b) Show that $A^n_{r,s}=A^{n+1}_{r,s}-(r+s)A^n_{r,s-1}$ for $r, s, n\geqslant 0$.

(c) Find an explicit formula for $A^n_{r,s}$. [*H. L. Krall, State College, Pennsylvania.*]

Solution: There is a combinatorial interpretation of this problem!

Divide a $1\times(2r+n)$ rectangle from the left into r 1×2 cells Z and n single squares E. Let $C^{n,r}_{u,s}$ denote the number of ways of placing u labeled balls of color U and s labeled balls of color S into the squares of the rectangle, at most one to a square, so that no cell Z is empty; we call this an (n,r,u,s)-arrangement.

There are $u!s!\binom{n+2r}{u}\binom{n+2r-u}{s}$ arrangements if we omit the condition that no cell be empty. The number of arrangements leaving empty the cells $j_1, j_2,\dots,j_i$ and possibly others $(1\leqslant i\leqslant r)$ is $u!s!\binom{n+2r-2i}{u}\binom{n+2r-u-2i}{s}$. By inclusion-exclusion we obtain

$$C^{n,r}_{u,s}=u!s!\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{n+2r-2i}{u}\binom{n+2r-u-2i}{s}. \tag{1}$$

If $s\geqslant 1$, then a change of the color of the ball with label s from S to U does not change the number of arrangements. Hence

$$C^{n,r}_{u+1,s-1}=C^{n,r}_{u,s}. \tag{2}$$

Furthermore,

$$C^{n,r}_{u,s}=2rC^{n,r-1}_{u,s-1}+(n+2r-u-s+1)C^{n,r}_{u,s-1} \tag{3}$$

for $r\geqslant 1$ and $s\geqslant 1$. We prove this as follows. From each (n,r,u,s)-arrangement we remove the ball of color S with label s. There are two cases.

(i) An $(n,r,u,s-1)$-arrangement results. Conversely, in this case each $(n,r,u,s-1)$-arrangement leads to an (n,r,u,s)-arrangement by putting an S-colored ball in one of the $n+2r-u-s+1$ free places.

(ii) One of the cells Z is empty. Removal of this Z results in an $(n,r-1,u,s-1)$-arrangement. Conversely, each $(n,r-1,u,s-1)$-arrangement leads to an (n,r,u,s)-arrangement by adding an empty cell Z (in r ways) and then placing an S-colored ball in one of its two places.

Since cases (i) and (ii) are mutually exclusive, (3) follows. Then use of (2) with $u\geqslant 1$ yields

$$C_{u,s}^{n,r} = 2rC_{u-1,s}^{n,r-1} + (n+2r-u-s+1)C_{u,s-1}^{n,r}. \tag{4}$$

Now set

$$A_{r,s}^{n} = \frac{C_{r,s}^{n,r}}{2^r \cdot r!}.$$

(NB: $u = r$.) Then (4) becomes

$$A_{r,s}^{n} = A_{r-1,s}^{n} + (n+r-s+1)A_{r,s-1}^{n},$$

which is the recursive formula of the problem. The initial values are easily verified. Therefore,

$$A_{r,s}^{n} = \frac{s!}{2^r}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{n+2r-2i}{r}\binom{n+r-2i}{s} \tag{5}$$

for $n \geqslant 0$ and $r + s > 0$.

Although (a) and (b) can be derived from (5) by simple transformations of combination symbols, combinatorial arguments like those used in the proof of (4) may be used instead.

J. C. BINZ
Universität Bern, Switzerland

Also solved by J. C. Binz (Switzerland, second solution) and the proposer.
Binz's second solution uses a combinatorial argument to derive the formal power series equality

$$\sum_{s=0}^{\infty} A_{r,s}^{n}\frac{x^s}{s!} = \frac{1}{2^r}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{n+2r-2i}{r}(1+x)^{n+r-2i}.$$

The proposer finds that

$$A_{r,s}^{n} = \sum_{i=0}^{\min(r,s)}\frac{(r+s)!\binom{n}{s-i}}{2^i \cdot i!(r-i)!}.$$

Relations (a) and (b) can be proved directly from the recursive definition by induction on $r+s$.

1147. *Also solved by R. C. Lyness (England), who noted that the same solution is obtained even if O is allowed to vary over the entire plane.* (Editor's misfiling.)

Three Triangle Constructions with Located Points — September 1982

1149. For each of the parts (a), (b), (c), separately, construct a triangle ABC, given in position the three points:

(a) O, M_a, I;
(b) O, H_a, T_a;
(c) M_a, H_a, I;

where O and I are the circumcenter and incenter of triangle ABC and M_a, H_a, and T_a are the points in which the median, the altitude, and the angle bisector to side a meet this side. (See the Note, "Triangle constructions with three located points", this MAGAZINE, v. 55 (1982) 227–230.) [*William Wernick, The Bronx, New York.*]

Solution: We use the notation $P(q)$ for the circle with center P and radius q. As usual, R denotes the circumradius and r the inradius of triangle ABC, and d denotes the distance OI.

(a) If $O \neq M_a$, then let L denote the line through M_a perpendicular to the line OM_a. (See FIGURE 1.) Since d is given and r is the perpendicular distance from I to L, Euler's formula $d^2 = R(R-2r)$ [see this MAGAZINE, v. 56 (1983) 325] can be solved for R:

$$R = r + \sqrt{r^2 + d^2},$$

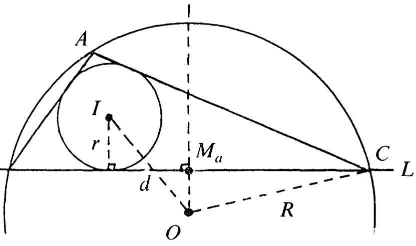


FIGURE 1. **Given O, M_a, I.**

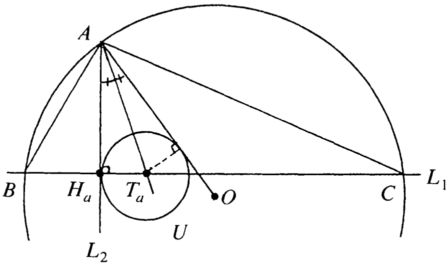


FIGURE 2 **Given O, H_a, T_a.**

and so R is constructible. Then B and C are located as the intersections of L with the circumcircle $O(R)$, and A is located as the intersection of the circumcircle with a tangent line (different from L) from B to the incircle $I(r)$.

If $O = M_a \neq I$, then the triangle ABC is not unique, and r may have any value such that $0 < r \leqslant d$. All such triangles have a right angle at A. The choice $r = d$ generates a triangle with internal angles $90°$, $45°$, $45°$, and the choice $r = d/\sqrt{2}$ generates a triangle with internal angles $90°$, $60°$, $30°$.

If $O = M_a = I$, then the triangle ABC degenerates to a point.

(b) Suppose first that $H_a \neq T_a$. Let L_1 denote the line H_aT_a, let L_2 denote the line through H_a perpendicular to L_1, and let U denote the circle $T_a(T_aH_a)$. (See FIGURE 2.) Since the interior angle bisector from a vertex bisects the angle between the altitude and the circumdiameter issued from that vertex [see David R. Davis, *Modern College Geometry*, p. 41], the vertex A is located as the intersection of L_2 with either tangent line from O to U. The points B and C are then located as the intersections of L_1 with the circumcircle $O(OA)$.

If $H_a = T_a \neq O$, then the triangle ABC is isosceles and not unique. Let L_2 denote the line OH_a and let L_1 denote the line through H_a perpendicular to L_2. Draw the circumcircle $O(R)$ with R greater than the distance OH_a. Then $O(R)$ intersects L_1 at B and C, and intersects L_2 at A.

If $H_a = T_a = O$, then take this common point to be the midpoint of the hypotenuse BC of an isosceles right triangle ABC.

(c) If $M_a \neq H_a$, let L denote the line M_aH_a, and let I_a denote the foot of the perpendicular from I to L. (See FIGURE 3.) Then r is the perpendicular distance from I to L, and I_a lies between H_a and M_a on L, since the internal bisector of angle A also bisects angle H_aAO. Since the distances AB and AC are unequal (otherwise we would have $H_a = M_a$), we choose the notation so that $AB < AC$. Let v, w, and x denote the distances M_aI_a, M_aH_a, and $\frac{1}{2}BC$, respectively. Then $w > v$. Now, since I lies on the bisectors of angles β and γ, we have

$$\tan\frac{\beta}{2} = \frac{r}{x - v} \quad \text{and} \quad \tan\frac{\gamma}{2} = \frac{r}{x + v}.$$

Also, the altitude AH_a is given by

$$(x - w)\tan\beta = AH_a = (x + w)\tan\gamma.$$

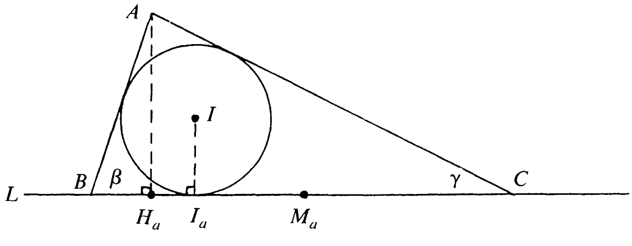


FIGURE 3 **Given M_a, H_a, I.**

Elimination of β and γ using the double-angle formula for the tangent yields

$$x^2 = v^2 + r^2\frac{w+v}{w-v}.$$

Since v, w, and r are already constructed, x can now be constructed. Then B and C are located as the intersections of L with the circle $M_a(x)$. Angles β and γ are found as twice the acute angles formed by L with the segments BI and CI, respectively. Now the desired triangle ABC is determined by two interior angles and the included side BC.

If $M_a = H_a \neq I$, then the triangle ABC is isosceles and not unique. Vertex A is chosen on the line H_aI produced beyond I so that $IA > IH_a$, and vertices B and C are located as the intersections of the tangents from A to the incircle $I(IH_a)$ with the line through H_a perpendicular to the line IH_a.

If $M_a = H_a = I$, then triangle ABC degenerates to a point.

J. M. STARK
Lamar University

Also solved by W. J. Blundon (Canada, (a) only), David DeKraker (student, (b) only), Ragnar Dybvik (Norway), L. Kuipers (Switzerland), Vania D. Mascioni (student, Switzerland, (b) only), Michael Vowe (Switzerland), Harry Zaremba, and the proposer. Several of the solutions contained formulas or constructions whose validity was not easy to check.

A Well-Known Combinatorial Identity **November 1982**

1154. Prove the combinatorial identity

$$\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{1}{(k+1)^2} = \frac{1}{n}H_n,\ n \geqslant 1,$$

where

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

[*Chico Problem Group, California State University.*]

Solution: The desired result is obtained by setting $f(x) = 1$ in the following generalization.

Let n be any positive integer and let $f(x)$ be a polynomial of degree less than or equal to n. Then

$$\sum_{k=0}^{n-1}\frac{(-1)^k\binom{n-1}{k}f(k+1)}{(k+1)^2} = \frac{1}{n}\left(f'(0) + f(0)\sum_{k=1}^{n}\frac{1}{k}\right).$$

Proof. For each integer $k, 0 \leqslant k \leqslant n$, we have

$$\lim_{x\to k}\frac{(-1)^n n! f(x)(x-k)}{x(x-1)\cdots(x-k)\cdots(x-n)} = (-1)^k\binom{n}{k}f(k),$$

and thus we get the partial fraction expansion

$$\frac{(-1)^n n! f(x)}{x(x-1)\cdots(x-n)} = \sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}f(k)}{x-k},$$

or

$$\sum_{k=1}^{n}\frac{(-1)^k\binom{n}{k}f(k)}{x-k}=\frac{(-1)^n n!f(x)}{x(x-1)\cdots(x-n)}-\frac{f(0)}{x}.$$

If we now let x approach 0, we obtain

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}f(k)}{k}=\lim_{x\to 0}\left(\frac{(-1)^n n!f(x)}{x(x-1)\cdots(x-n)}-\frac{f(0)}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{(-1)^n n!}{(x-1)\cdots(x-n)}\cdot\frac{f(x)-f(0)}{x}+f(0)\cdot\frac{(-1)^n n!-(x-1)\cdots(x-n)}{x(x-1)\cdots(x-n)}\right)$$

$$=f'(0)+f(0)\sum_{k=1}^{n}\frac{1}{k},$$

and, consequently, after reindexing the left side and using the identity

$$\binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{k},$$

we get the result.

In a similar way we can prove the more general result:

Let m and n be any integers such that $m > n \geqslant 1$, and let $f(x)$ be a polynomial of degree less than or equal to m. Then

$$\sum_{k=0}^{n-1}\frac{(-1)^k\binom{n-1}{k}f(k+1)}{(k+1)^2}$$

$$=\frac{1}{n}\left(f'(0)+f(0)\sum_{k=1}^{n}\frac{1}{k}\right)+(-1)^{n-1}(n-1)!\sum_{k=1}^{m-n}\frac{f^{(n+k)}(0)}{(n+k)!}S(n+k-1,n),$$

where the $S(m,k)$ are Stirling numbers of the second kind.

ARMEL MERCIER
Université du Québec à Chicoutimi

Also solved by Donald Batman, Erhard Braune (Austria), Charles Chouteau, Curtis Cooper, Alan Edelman, Nick Franceschine, Enzo R. Gentile (Argentina), Leslie V. Glickman (England), Robert E. Greenwood, Hexlovers (Brazil), James C. Hickman, Michael Josephy (Costa Rica), D. G. Kabe (Canada), Hans Kappus (Switzerland), Murray S. Klamkin (Canada), Benjamin G. Klein, Ruth Koelle, L. Kuipers (Switzerland), Sai Chong Kwok, Robert A. Leslie, Vania D. Mascioni (student, Switzerland), Peter L. Montgomery, Roger B. Nelsen (two solutions), Stanley Rabinowitz, M. Ratnaprabhu & T. Narasimham (India), St. Olaf College Problem Solving Group, Morris S. Samberg, Leo Y. Satiadi, Donald R. Schuette, Heinz-Jürgen Seiffert (student, West Germany; five solutions!), J. M. Stark, R. J. Stroeker (The Netherlands), J. Suck (West Germany), Vis Upatisringa, L. Van Hamme (Belgium), S. K. Venkatesan (India), David Vopicka, Michael Vowe (Switzerland), Thomas C. Wales, Max L. Warshauer, and the proposer (two solutions).

Many solvers mentioned that the result is easily obtainable from the well-known identity

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}k^{-1}=H_n,$$

and supplied references, among which are: Pólya-Szegö, problem I.38; *Monthly* problems 4130 (v. 52 (1945), 527–529) and E864 (v. 57 (1950), 38); this MAGAZINE problem 335 (v. 32 (1958), 107–108); example 3 on p. 4 of Riordan's and formula 1.45 of Gould's *Combinatorial identities*; and Hietala and Winter, "Note on a combinatorial identity," this MAGAZINE, v. 38 (1965), 149–151. Most of these give further references.

Hexlovers proved the following generalization. Let

$$S_{p,n}=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(k+1)^{-p}$$

for p real and $n\in\{1,2,3,\dots\}$. Then $S_{p,n}$ is equal to 1 if $n=1$ or $p=1$, to 0 if $p=0$ and $n\geqslant 2$, and to $S_{p,n-1}+n^{-1}S_{p-1,n}$ otherwise. Greenwood also generalized the problem.

Constant Tangent Lengths — November 1982

1155. A plane intersects a sphere forming two spherical segments. Let S be one of these segments and let A be the point of the sphere furthest from the segment S. Prove that the length of the tangent from A to a variable sphere inscribed in the segment S is constant. [*Stanley Rabinowitz, Merrimack, New Hampshire.*]

Solution: Designate the given sphere and cutting plane by s and p respectively, an arbitrary sphere inscribed in S by t, and the sphere having center A and passing through the intersection of s and p by u. Inversion in u maps p onto s and therefore t onto itself. It follows that t is orthogonal to u and the tangent from A to t is equal to the radius of u.

HOWARD EVES
University of Maine

Also solved by Anders Bager (Denmark), Jordi Dou (Spain), Ragnar Dybvik (Norway), Alan Edelman, Herta T. Freitag, Michael Goldberg, Hans Kappus (Switzerland), Murray S. Klamkin (Canada), Stanley Rabinowitz, L. Kuipers (Switzerland), Henry S. Lieberman, Robert Patenaude, James Propp (student, England), J. M. Stark, George Tsintsifas (Greece), Michael Vowe (Switzerland), Harry Zaremba, and the proposer.

Bager, Klamkin, and Propp considered the extension of the problem to n dimensions. In Dou's solution, the sphere t may be externally tangent to s. Dou also generalized the problem by replacing the plane p by a second sphere s'; then the point A is replaced by the center of (direct or inverse) similitude of s and s'. (See FIGURES.)

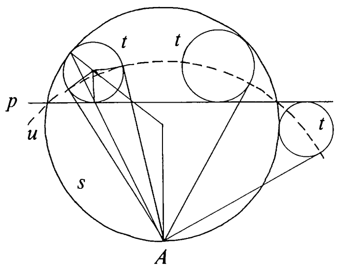


FIGURE 1

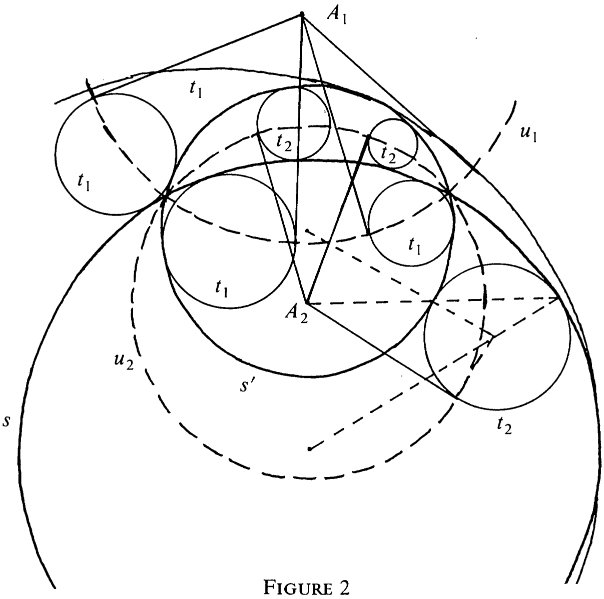


FIGURE 2

A Two-Triangle Inequality — November 1982

1156. Let a, b, c, and F be the three sides and the area of triangle ABC, and let a', b', c', and F' be the corresponding quantities for triangle $A'B'C'$. Show that

$$a'(-a+b+c)+b'(a-b+c)+c'(a+b-c) \geqslant \sqrt{48FF'},$$

with equality if and only if both triangles are equilateral. [*Gao Ling, Chongqing, China.*]

Solution: Replace each triangle by a similar triangle so that $a+b+c=a'+b'+c'=2$. By Heron's formula, we have

$$F^2=(1-a)(1-b)(1-c) \quad \text{and} \quad F'^2=(1-a')(1-b')(1-c').$$

Define $u:=1-a, v:=1-b, w:=1-c$, and similarly u', v', w'. Then $0<u, v, w, u', v', w'<1$ and $u+v+w=u'+v'+w'=1$. The inequality to be proved then becomes (surprise!)

$$(144\,uvwu'v'w')^{1/4}+uu'+vv'+ww'\leqslant 1,$$

which is the result of applying the Cauchy-Schwarz inequality to

$$\sqrt{12\,uvw}+u^2+v^2+w^2\leqslant 1 \tag{1}$$

and the analogous inequality for u', v', w'. (Equality in the Cauchy-Schwarz inequality occurs if and only if ABC and $A'B'C'$ are similar.) Now (1) follows from $\sqrt{3\,uvw}\leqslant uv+vw+wu$, which in turn follows from the easily proved inequality

$$(u+v+w)\,uvw\leqslant u^2v^2+v^2w^2+w^2u^2,$$

with equality if and only if $u=v=w=\frac{1}{3}$, i.e., triangle ABC is equilateral.

VANIA D. MASCIONI, student
E. T. H. Zürich, Switzerland

Also solved by Erhard Braune (Austria), Leonard D. Goldstone, Murray S. Klamkin (Canada), R. J. Stroeker (the Netherlands, two solutions), George Tsintsifas (Greece), Michael Vowe (Switzerland), and the proposer (three solutions).

Most of the solutions used formulas 10.3 and 10.8 from O. Bottema et al., *Geometric inequalities*; others used formula 4.9, 5.7 or 14.1. Two direct solutions obtained the result only for some permutation of a', b', c', for fixed a, b, c.

A Tipsy Wineglass **November 1982**

1157. The interior surface of a wine glass is a right circular cone. The glass contains some wine and is tilted so that the wine-to-air interface is an ellipse of eccentricity e and is at right angles to a generator of the cone.

Prove that the area of the ellipse is e times the area of that part of the curved surface of the cone which is in contact with the wine. [*R. C. Lyness, Southwold, Suffolk, England.*]

Solution: Let rectangular (x, y, z) coordinates be taken in such a way that the vertex of the right circular cone is at the origin and the positive z-axis is the axis of that nappe containing the ellipse. Denote by α the acute angle between a generator of the cone and the axis of the cone, and let β be the acute angle between the axis of the cone and the plane of the wine-to-air interface.

Let a be the semimajor axis of the ellipse, and let c be half the distance between its foci. Denote by r the radius of the smaller Dandelin sphere inscribed between cone and plane, and suppose that the (x, z)-plane contains in its first quadrant that generator of the cone which is at right angles to the plane of the wine-to-air interface.

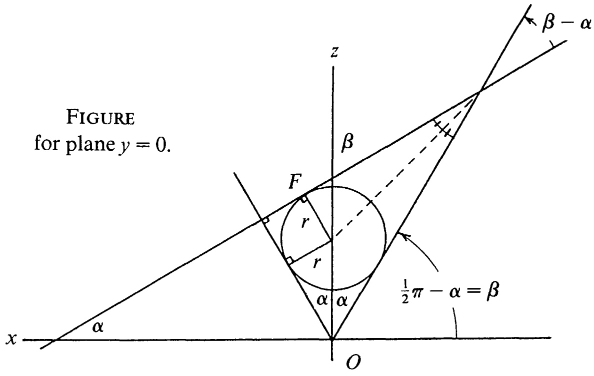
Now consider right triangle geometry in the $y=0$ plane as shown in the FIGURE. The circle in the FIGURE is a great circle of the smaller Dandelin sphere and intersects the wine-to-air plane at a focus F of the ellipse. The plane of the FIGURE, namely $y=0$, contains the major axis of the ellipse.

Right triangle geometry gives

$$2a=r+r\cot[(\beta-\alpha)/2]. \tag{1}$$

Using $r=a-c$, $\alpha+\beta=\pi/2$ to eliminate r and β from (1) gives a result which can be written

$$c/a=\tan\alpha, \tag{2}$$



where $c/a = e$, the eccentricity of the ellipse.

The curved surface A_L in contact with the wine and the area of the ellipse A_E project orthogonally onto a common area A in the (x, y)-plane. Each area element of A_L makes the angle $\beta = \frac{1}{2}\pi - \alpha$ with the (x, y)-plane, and the plane of A_E makes angle α with the (x, y)-plane. Hence

$$A_E = A/\cos\alpha, \quad A_L = A/\cos\left(\tfrac{1}{2}\pi - \alpha\right),$$

from which it follows by use of (2) that $A_E = A_L \tan\alpha = eA_L$.

J. M. STARK
Lamar University

Also solved by Jordi Dou (Spain), Hans Kappus (Switzerland), Harold Slater & John Putz, Robert S. Stacy (West Germany), Robert L. Young, Harry Zaremba, and the proposer. Partial solution by M. Ratnaprabhu & T. Narasimham (India). There was one incorrect solution.

A Pseudo-Fibonacci Limit **November 1982**

1158. Set $a_0 = 1$ and for $n > 1$, $a_n = a_{n'} + a_{n''}$, where $n' = [n/2]$ and $n'' = [n/3]$. Find $\lim_{n\to\infty} a_n/n$. [*Anon, Erewhon-upon-Spanish River.*]

Solution (Editor's composite): Let r be the solution for x of the equation $2^{-x} + 3^{-x} = 1$ in the interval $(0,1)$. (Such a unique solution exists, since the left side depends continuously and monotonically on x, and has the values 2 and 5/6 for $x = 0$ and $x = 1$, respectively; in fact, $r \approx 0.7879$.) We show by induction that $a_n \leqslant 3n^r$ for $n \geqslant 1$. This is obvious for $1 \leqslant n \leqslant 2$. If $n \geqslant 3$, then $n > n' \geqslant n'' \geqslant 1$, and assuming the inequality for all positive integral values less than n, we have

$$a_n = a_{n'} + a_{n''} \leqslant 3(n')^r + 3(n'')^r \leqslant 3\left(\frac{n}{2}\right)^r + 3\left(\frac{n}{3}\right)^r = 3n^r.$$

Since $0 \leqslant a_n/n \leqslant 3n^{r-1}$ for all $n \geqslant 1$, the required limit exists and has the value 0.

NICK FRANCESCHINE
Sebastopol, California
and, independently,
PETER L. MONTGOMERY
System Development Corporation
Santa Monica, California

Also solved by Elise Andrews, Duane M. Broline, Stephen D. Bronn, G. A. Heuer, David Iny, Michael Josephy (Costa Rica), L. Kuipers (Switzerland), Örsett Künt (Brazil), Petr Lisoněk (Czechoslovakia), Vania D. Mascioni (student, Switzerland), Robert Patenaude, James Propp (student, England), Daniel A. Rawsthorne, St. Olaf Problem Solving Group, Heinz-Jürgen Seiffert (student, West Germany), James T. Smith, L. Van Hamme (Belgium), and the proposer.

Andrews caught the editors's error in the problem statement ("for $n>1$" should be "for $n \geqslant 1$"). In most of the solutions it was shown by induction on k that $a_n \leqslant bnp^k$ when $n \geqslant cq^k$, for appropriate constants b, c, p, and q, with an initial induction on n for the case $k=0$.

Künt and Patenaude stated generalizations. Patenaude writes: "One expects $a(n)/n \equiv a_n/n \to 0$ as $n \to \infty$ for similar sequences. Suppose

$$0<r\equiv r_1 \leqslant r_2 \leqslant \cdots \leqslant r_m \quad \text{with } r_1+\cdots+r_m \equiv s<1,$$

and let $f\colon R^+ \to \{0,1,\dots\}$ satisfy $f(x)\leqslant x$. Fix $a(0)\geqslant 0$ and define

$$a(n)=a(f(r_1 n))+\cdots+a(f(r_m n))$$

for $n \geqslant 1$. Then $0 \leqslant a(n) \leqslant Ms^k n$, where $M=\max\{a(n)/n : 1 \leqslant n < 1/r\}$ and $k=[\log_{1/r} n]$. Here $m=2$, $r_1=1/3$, $r_2=1/2$, and $f(x)=[x]$." Compare also proposal 1185, this issue.

A Stacking Problem — November 1982

1159. Phone books, n in number, are kept in a stack. The probability that the book numbered i (where $1 \leqslant i \leqslant n$) is consulted for a given phone call is $p_i>0$, where $\Sigma_{i=1}^n p_i=1$. After a book is used, it is placed at the top of the stack. Assuming that the calls are independent and evenly spaced, and that the system has been employed indefinitely far into the past, let P be the probability that, right now, each book is in its proper place, the book numbered i being ith from the top for $1 \leqslant i \leqslant n$. Unfixing the p_i's, find the least upper bound of P. [*James Propp, student, Cambridge University.*]

Solution I: We assume none of the books have been stolen by vandals. The probability that book 1 is in place is the probability that the last phone call referenced book 1, namely p_1. The probability that book 2 is in place (given that book 1 is in place) is

$$p_2+p_2p_1+p_2p_1^2+\cdots=p_2/(1-p_1).$$

Continuing, we find

$$P=p_1\cdot\frac{p_2}{1-p_1}\frac{p_3}{1-p_1-p_2}\cdots\frac{p_n}{1-p_1-p_2-\cdots-p_{n-1}}. \tag{1}$$

This probability can be arbitrarily close to 1; let

$$p_1=1-q,\quad p_2=q-q^2,\quad \dots,\quad p_n=q^{n-1}$$

to get $P=(1-q)^{n-1}$, where q is close to 0.

PETER L. MONTGOMERY
System Development Corporation
Santa Monica, California

Solution II: Formula (1) works if all the p_i's are nonzero. If $p_i=0$ for some i, then $P=0$ unless for some k we have $p_i>0$ for $1 \leqslant i \leqslant k$ and $p_i=0$ for $k<i\leqslant n$. In this case, we get

$$P=p_1\cdot\frac{p_2}{1-p_1}\cdot\frac{p_3}{1-p_1-p_2}\cdots\frac{p_k}{1-p_1-p_2-\cdots-p_{k-1}}\cdot\frac{1}{(n-k)!}$$

because the arrangement of books $k+1$ through n depends only on their initial configuration, which has a probability of $1/(n-k)!$ of being in order if all initial configurations are equally likely.

JAMES PROPP, student
Cambridge University

Also solved by Jordi Dou (Spain).

REVIEWS

Paul J. Campbell, Editor
Beloit College

Assitant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Blair, Douglas H., and Pollak, Robert A., *Rational collective choice*, Scientific American 249:2 (August 1983) 88-95, 128.

Extensive analysis of the axiomatics of voting systems, beginning with Arrow's "five intuitively appealing axioms" (which cannot be satisfied together except by a dictatorship) and carefully examining weaker alternatives to them. The conclusion offers little comfort: "...the opportunities for improvement [of voting methods] are severely limited. Stark compromises are inevitable."

Walker, Jearl, *The amateur scientist: In which simple equations show whether a knot will hold or slip*, Scientific American 249:2 (August 1983) 120-128.

Recounts the impressive theory of B. F. Bayman (Minnesota), who has modeled the static forces in knots. Some hitches are more secure than others, and Bayman's theory uses determinants and systems of inequalities to tell just how strong a knot is.

Gardner, Martin, *Mathematical games: tasks you cannot help finishing, no matter how hard you try to block finishing them*, Scientific American 249:2 (August 1983) 12-21, 128.

A surprise! Gardner is back for a guest appearance, examining seemingly endless tasks that surprisingly must terminate finitely. One, Bulgarian solitaire, leads to mathematical partition theory, which in turn has become enormously useful in particle physics. Now: how soon will a new volume be published of some of the 200 or so uncollected Gardner columns?

Gardner, Martin, *Mathematical games: the topology of knots, plus the results of Douglas Hofstadter's Luring Lottery*, Scientific American 249:3 (September 1983) 18-28, 202.

Gardner returns to the subject of knots, Hofstadter to the superrationality (a form of the Kantian imperative) of his June column. With best wishes for his readers, Hofstadter announces the close of his term as "Metamagical Themas" columnist.

Walker, Jearl, *The amateur scientist: the physics of the follow, the draw and the massé (in billiards and pool)*, Scientific American 249:1 (July 1983) 124-128.

Excellent exposition of the qualitative classical mechanics (friction, spin) underlying standard and trick shots in pool.

Hayes, Brian, *Computer recreations: Introducing a department concerned with the pleasures of computation*, Scientific American 249:4 (October 1983) 22-36, 170.

Start of a new era in Scientific American: a new column on Computer Recreations, to replace Hofstadter's Metamagical Themas. This first column considers electronic spreadsheets and their potential for modeling systems of cellular automata. The game of Life is easy to do, and the Ising model of ferromagnetism is harder (randomization is involved), and percolation models are left to the ingenuity of the readership.

Kolata, Gina, *Number theory problem is solved*, Science 221 (22 July 1983) 349-350.

Gerd Faltings (Wuppertal U.) has proved the 60-year-old Mordell conjecture: "algebraic curves whose associated topological surface has two or more holes have only a finite number of rational solutions in any given number field." Comments Spencer Bloch (Chicago): "... at least in number theory, this is the theorem of the century."

Dembart, Lee, *Mathematician proves basic theory adds up; work hailed as milestone*, Los Angeles Times (14 July 1983) 3, 18.

Popular account of confirmation of the Mordell conjecture ("most equations higher than the third degree have only a finite number of rational solutions"). Hailed as a magnificent achievement, the result by Gerd Faltings (Wuppertal U.) also offers support for Fermat's "Last Theorem," since the possibility is ruled out that for some n there could be infinitely many solutions.

Dembart, Lee, *New codes may be key to atom test ban treaty*, Los Angeles Times (5 July 1983) 1, 12.

Public-key cryptosystems could be used in automatic seismic installations to guarantee untampered data in monitoring for atomic weapon tests. Such a system was the key to the nuclear test ban treaty pursued by the Carter administration, a goal abandoned by the Reagan administration.

Billingsley, Patrick, *The singular function of bold play: Gambling theory provides a natural example of a function with anomalous differentiability properties*, American Scientist 71 (July-August 1983) 392-397.

Gambling theory provides a natural example of an "anomalous" function. The graph of the probability of success of a bold gambler is both self-similar and singular (continuous and increasing but with 0 derivative almost everywhere).

Gleick, James, *Exploring the labyrinth of the mind*, New York Times Magazine (21 August 1983) 1, 23-27, 83, 86-87, 100.

Douglas Hofstadter provoked in Gödel, Escher, Bach a renaissance in philosophy of the mind. His bottom-up, pattern-intuition approach to creating artificial intelligence, however, flies in the face of the current AI emphasis on "expert systems" for the marketplace. Hofstadter rejects their inflexibility as unintelligent; he in turn is faulted for not putting forward working programs. Says Marvin Minsky, "50 years from now, they'll say he was on the right track... . Hofstadter's philosophical ideas on how the mind works are just about the best in the world today."

Dembart, Lee, *Scientists find new high in prime numbers game*, Los Angeles Times (23 September 1983) 1, 21.

$2^{132049}-1$: The newest Mersenne prime, the 29th, was found by David Slowinski (Cray Research) as a way of advertising the new Cray XMP ($9 million). "It's like racing computers," he says.

The Birth of Computers, New Scientist 99 (15 September 1983) 778-791 + poster.

Three articles plus a poster, on "catching up with Babbage," details of his designs, Turing's computer building, and the history of computing. Notable are the analyses of why Babbage's Analytical Engine was never completed (*not* because of unattainable manufacture, as often alleged), and why Turing's national computer plan was not pursued. There were certainly exogenous reasons, but prominent was each man's unwillingness and inability to deal with the full complexity of interaction with the human social and political structure. (Hardware freaks and computer evangelists, take heed!)

Ford, Joseph, *How random is a coin toss?*, Physics Today 36:4 (April 1983) 40-47.

How can a sequence be both truly random and strictly deterministic? What would mathematics be like without "incalculable" numbers, those which cannot be calculated by any finite algorithm? What are the implications for physics of the loss of infinite computational and observational precision? This thought-provoking article starts with examining differences between orderly and chaotic behavior in solutions to dynamical problems and concludes with profound speculation on the future of physics.

Kac, Mark, *Marginalia: what is random?*, American Scientist 71:4 (July-August 1983) 405-406.

"Let me then state categorically that there is no way to tell with any degree of confidence whether a sequence of *H*'s and *T*'s has been by tosses of a coin or by an arithmetical procedure... . Fortunately, the upper reaches of a science are as insensitive to such basic questions as they are to all sorts of other philosophical concerns." (Beware--the preponderance of space devoted to the prominence of the normal distribution may accidentally reinforce the mistaken notion of many students that "random" means "normally distributed.")

Morris, Scot, *Games: The hollow Earth: a maddening theory that can't be disproved*, Omni 5:11 (August 1983) 128-129.

"The earth is a hollow sphere, and we live inside it." What a crazy proposition! So crazy, a cult grew up around it; and so crazy, a little fiddling with inversive geometry turns it into a proposition virtually impossible to refute.

Kirkpatrick, S., *et al.*, *Optimization by simulated annealing*, Science 220 (13 May 1983) 671-680.

The authors have realized a striking analogy between statistical mechanics and combinatorial optimization. They apply the Metropolis algorithm (for approximate simulation of the behavior of a many-body system at a finite temperature) to the travelling salesman problem and others. The algorithm finds the low-temperature state of a system by "annealing" it: "melt" the system at a higher temperature, then lower the temperature by slow stages until the system "freezes" and no further changes occur. The key idea in the analogy is to introduce a "temperature" in the optimization problem. Applications are given to computer chip design, placement of chips on layers, and wire-routing among circuits.

Nahin, Paul J., *Oliver Heaviside: genius and curmudgeon*, IEEE Spectrum (July 1983) 63-69.

Students of physics or advanced calculus hear briefly of Heaviside's contributions. Apart from noting his contributions to telephony and explaining his unpopularity with his contemporaries (mathematicians and physicists alike), this article offers some details of this hermit genius's life: self-educated school dropout, telegrapher, independent scientist (after retirement at age 24!).

Cook, Stephen A., *An overview of computational complexity*, Communications ACM 26 (1983) 401-408.

Turing Award lecture giving history and current frontiers of computational complexity, including consideration of probabilistic and parallel computation.

Bartholdi, John J., III, *et al., A minimal technology routing system for meals on wheels*, Interfaces 13:3 (June 1983) 1-8.

Successful implementation of a novel routing system based on a new travelling salesman heuristic, for which a spacefilling curve serves as prototype. Excellent illustration of no-resources operations research: 13% improvement, cost less than $50, no computer needed.

Dixon, Robert, *Geometry comes up to date*, New Scientist 98 (5 May 1983) 302-305.

Isometries and inversions are described and illustrated; and the ease of their use with computer graphics is emphasized by the author, a geometer at the Royal College of Art.

Schneer, Cecil J., *The Renaissance background to crystallography*, American Scientist (May-June 1983) 254-263.

The history of crystallography is traced: from the efforts of the Neopythagoreans of the 15th and 16th centuries (particularly Kepler) to seek fundamental numerical harmony in the universe, through the empirical approach of Buffon, to Haüy's formulation of the agenda for modern crystallography.

Scarf, Herbert E., *Fixed-point theorems and economic analysis*, American Scientist 71 (May-June 1983) 289-296.

Offers an example of the relevance of combinatorial mathematics to applied mathematics, as Sperner's lemma on simplices is used to give a nonconstructive proof of the Brouwer fixed-point theorem and later to spawn constructive algorithms.

Chandler, Bruce, and Magnus, Wilhelm, The History of Combinatorial Group Theory: A Case Study in the History of Ideas, Springer-Verlag, 1982; viii + 234 pp.

The theory of groups defined by generators and relations is taken by the authors as a case study of the development of an independent discipline from the attempted solutions of problems in other areas. The authors concentrate on phenomena of the transmission of ideas: the modes of communication that knit together the specialty, geographical distribution of research, and the effects of migration. Their concepts of *concentration, storage,* and *streamlining* will long be useful in probing the organization of mathematics. This is a book which can be enjoyed by any mathematician or graduate student interested in where mathematical concepts come from and how mathematical ideas develop.

Flagg, Graham, Numbers: Their History and Meaning, Schocken, 1983; x + 295 pp, $14.50.

This easy-reading book collects together ideas and techniques associated with numbers over the centuries. This history of counting, number representations, various paper and pencil algorithms, the rise of algebra and trigonometry, and recreations with numbers are all treated.

Le Lionnais, Francois, Les Nombres Remarquables, Hermann, 1983; 158 pp, 86F (P).

A Michelin guide to numbers and their properties, complete with *, **, and *** ratings. Well-researched; a "must" for numerical nugget collectors.

Rademacher, H., Higher Mathematics from an Elementary Point of View, (ed.), D. Goldfeld, Birkhäuser, 1983; iii + 138 pp, $19.95.

Hans Rademacher (1892-1969) gives us still another gift: a previously unpublished book of lectures given at Stanford in 1947. The editor has inserted notes sparingly to bring the account up to date. Rademacher treats some topics from elementary number theory, then goes on to define Ford circles, discuss the modular group and functions on groups, and ends with a chapter on linkages. The spirit is reminiscent of his famous The Enjoyment of Mathematics, and we can be happy to have this to savor, too.

Rapoport, Anatol, Mathematical Models in the Social and Behavioral Sciences, Wiley, 1983; xii + 507 pp, $49.95.

"This book is offered as an attempt to demonstrate the *integrative* function of the mathematical mode of cognition... . Thus the goal...is that of restructuring habits of thinking about social phenomena. Similar mathematical models can represent widely different contents in the behavioral sciences." This book, organized not by content areas but by classes of models, includes both fruitful and sterile models. It is the comments on distinguishing them, and in the challenging final chapter on "social antecedents and consequences of mathematization," that make this book tower over the cookbook texts on modeling.

Lucas, William F., (ed.), Modules in Applied Mathematics, 4 vols., Springer-Verlag, 1983, $28 each.

Vol. 1: Braun, William, *et al.* (eds.), Differential Equation Models, xix + 380 pp.
Vol. 2: Brams, Steven J., *et al.* (eds.), Political and Related Models, xx + 396 pp.
Vol. 3: Lucas, William F., *et al.* (eds.), Discrete and System Models, xx + 353 pp.
Vol. 4: Marcus-Roberts, Helen, and Thompson, Maynard, Life Science Models, xx + 366 pp.

An absolute cornucopia of case studies and examples in modeling! Collected here are almost all of the modules resulting from two MAA projects: the 1976 summer workshop at Cornell, and the CUPM project on case studies in applied mathematics. A total of 63 contributions are included. Because of the delayed publication, some will be familiar from having appeared elsewhere (e.g., several modules by Braun are incorporated into his differential equations text). Stated prerequisites vary from high-school algebra on up, and time from an hour to six weeks. Exercises are included in each module. No college mathematics library should be without these books.

Cipra, Barry, Misteaks...and How to Find Them Before the Teacher Does...A Calculus Supplement, Birkhäuser, 1983; xiii + 69 pp. $4.95.

This book is full of mostly excellent advice on how to check answers in calculus for reasonableness, and it uses the effective device of giving as exercises solved calculus problems whose faults the reader is invited to find. However, in his attempt to reach students, the author occasionally hits a new low in sophomoric advice: "When possible, use symbols which will confuse anyone trying to grade your paper, then complain when they grade it wrong"; "Fudge and forget it!"

Schlefer, Jonathan, *Math art, a special report: predestined sculptures*, Technology Review 86:4 (May-June 1983) 54-60, 87.

Sculptor Morton C. Bradley, Jr., makes colorful pieces based on stellations of the regular polyhedra, and this article includes an interview with him.

NEWS & LETTERS

CHAUVENET PRIZE

R. Arthur Knoebel, of New Mexico State University, is the recipient of the 1984 Chauvenet Prize for his paper "Exponentials Reiterated" (*American Math. Monthly* 88 (1981) 235-252). The $500 prize, awarded at the Jan. 1984 MAA meeting, recognizes a noteworthy expository or survey paper published in a North American Journal. Knoebel's paper was earlier chosen as a Lester R. Ford prize winner.

POLYHEDRA CONFERENCE

SHAPING SPACE, an interdisciplinary conference on polyhedra, will be held at Smith College on April 6-8, 1984. Through lectures, exhibits, and workshops, the conference will address several related questions, including geometry in scientific thought, the role of tactile and visual perception in learning mathematics, current research and unsolved problems in the theory of polyhedra, and implications for high school and college geometry curricula.

In addition, there will be four sessions on the following topics:

I. Model-building as hobby and profession
II. Applications of polyhedra
III. Theory of polyhedra
IV. Unsolved problems

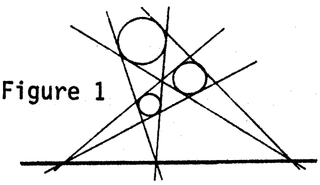
For further information write to:
Marjorie Senechal
Clark Science Center, Smith College
Northampton, MA 01063

3-CIRCLE THEOREM MADE VISIBLE

The Three-Circle theorem (see J. McCleary, "An Application of Desargues' Theorem", this *Magazine*, Sept. 1982, 233-235) has a spatial interpretation so that it can be read off directly from a figure.

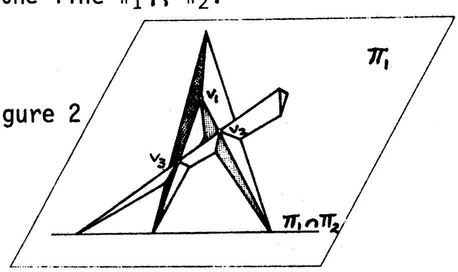
THREE-CIRCLE THEOREM: *Given three circles, none contained within another, and with distinct radii, then the three intersection points of common external tangents of pairs of circles lie on one straight line* (Figure 1).

Figure 1



Proof: Consider the circles (in a plane π_1) as bases of cones with angle of slope 45°, say. As shown in Figure 2, these three cones determine three "embankments" such that the base lines (in π_1) are tangents of the given circles. The vertices v_1, v_2, v_3 of the cones determine a plane π_2 which contains the ridges of the embankments. The three points where the embankments emerge from π_1 are the intersections of common external tangents. They lie on the line $\pi_1 \cap \pi_2$.

Figure 2



This proof works equally well if the circles intersect each other or if one circle lies in the convex hull of the other two circles. The common tangents may also cross each other: one has to orient the circles and to draw the tangents so that they respect the orientation (compare Blaschke [1], II.39). Moreover, the circles can be replaced by convex domains which arise from a given convex domain by translation and similarity; this generalized Three-Circle theorem belongs to affine geometry. A picture of the taxicab metric version can be seen in [2].

[1] W. Blaschke, Analytische Geometrie, Birkhaüser, 2nd ed. 1954.

[2] Math. Intellig., 5 (1983) 44-45.

Erhard Heil
Technische Hochschule
6100 Darmstadt, W. Germany

MATRICES OF BINOMIAL TYPE

Interesting polynomial-entried matrices were recently presented by Dan Kalman (this *Magazine*, Jan. 1983, 23-25). These matrices can be placed in a larger context of unimodular, lower triangular matrices $P(x)$ the entries of which are "polynomials of binomial type". Let $P(x)$, $-\infty < x < \infty$, be the $n\times n$ ($n\geq 2$) matrix $P(x)=(p_{ij})$ where $p_{ij}= P_{i-j}(x)$ when $i\geq j$, and $p_{ij}=0$ when $i<j$. The matrix identity

$$P(x)P(y)=P(x+y)$$

is equivalent to the set of equations

$$P_r(x+y) = \sum_{m=0}^{r} P_{r-m}(x)P_m(y), \quad r=0,1,2\ldots$$

which, in turn, characterizes polynomials of binomial type (R. Mullin and G.-C. Rota, in Graph Theory and its Applications, Academic Press, 1970, 167-213). The matrix presented by Kalman is slightly different from (p_{ij}) having entries $p'_{ij}=c_{ij}p_{ij}$, c_{ij} being constants. The entries p'_{ij} are thus "Sheffer polynomials" (see e.g. G.-C. Rota *et al*, J. Math. Anal. Appl. 42 (1973) 684-760).

Abraham Ungar
Rhodes University
Grahamstown
South Africa

MORE VOTING PARADOXES

In "Paradoxes of Preferential Voting" (this *Magazine*, September 1983, 207-214), Fishburn and Brams have shown that in Hare's preferential voting system two types of no-show paradoxes occur:

1. The addition of a ballot with candidate X ranked last may change the winner from another candidate to X.

2. The addition of a ballot with candidate Y ranked first may change the winner from Y to another candidate.

We note that both of these no-show paradoxes can result from a single additional cast ballot. Consider the following preference profile of five candidates A, B, C, D and E for a four-member district with 99 voters.

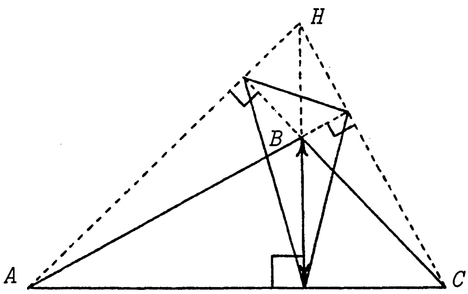
No. of Voters	Order of Preference 1st	2nd	3rd	4th	5th
60	D	B	A	C	E
5	A	E	C	B	D
15	A	C	E	B	D
19	E	C	A	B	D

Here under the usual procedure in Hare's system, the vote quota needed for a candidate to win is the smallest integer greater than 99/(4+1) which is 20 and the winning candidates are A, B, C and D. However, if in the above preference profile one adds a voter whose preferences are in order C, A, B, D, E, then the number of voters increases to 100 and hence the vote quota is now the smallest integer greater than 100/(4+1), which is 21. Thus the winning candidates now are A, B, D and E. In this case, the addition of a voter whose preference is C and last preference is E turns C from a winner to a loser and E from a loser to a winner. This is more unsatisfactory than two paradoxes occurring separately.

An important shortcoming of Hare's system which was not pointed out by Brams and Fishburn is that Hare's system under certain circumstances simply fails to give the required number of winners. For instance, if a committee of 24 members is to elect a subcommittee of seven, then the vote quota is the smallest integer greater than 24/(7+1), which is 4. But if the vote quota is 4, then obviously all the votes get exhausted by electing only six candidates and there is no way to elect the seventh.

Dipankar Ray
Calcutta University
Calcutta, India

TRIANGLE RHYME REVISITED



The orthic (as the name was used
  in Dwight Paine's charming rhyme)[1]
It need not have the properties
  he gave it at that time.
The problem is just this that when
  B's angle is obtuse,
The orthic then, for shortest path,
  will simply be no use.
Instead of which the shortest path
  to touch the sides (all three)
Is just a simple cycle on
  the altitude at B.
The orthic's angle bisectors,
  as can be seen (and proved),
Will meet at B, that is to say,
  from H somewhat removed.

Professor Paine was elsewise right
  in everything he said.
I'm merely writing to be sure
  that no one was misled.

[1] "Triangle Rhyme", this *Magazine*, Sept. 1983, 235-239.

Joseph McHugh
LaSalle College
Philadelphia, Pa.

OLYMPIAD PROBLEMS AND SOLUTIONS

The six problems from the International Math. Olympiad which challenged student teams from 32 countries last July are presented here to challenge our readers. Solutions will appear in our next issue.

The solutions to the USAMO and Canadian Olympiad problems which follow have been especially prepared for publication in this *Magazine* by Loren Larson and Bruce Hanson, St. Olaf College.

24th INTERNATIONAL MATHEMATICAL OLYMPIAD PARIS, JULY 6-7, 1983

1. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

   (i) $f(xf(y)) = yf(x)$ for all positive x, y;
   (ii) $f(x) \to 0$ as $x \to +\infty$.

2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centres O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that the angles O_1AO_2 and M_1AM_2 are equal.

3. Let a, b, c be positive integers, no two of which have a common divisor greater than 1. Show that
$$2abc - ab - bc - ca$$
is the largest integer which cannot be expressed in the form
$$xbc + yca + zab$$
where x, y, z are non-negative integers.

4. Let ABC be an equilateral triangle, and E be the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

5. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.

6. Let a, b, and c be the lengths of the sides of a triangle. Prove that
$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$
Determine when equality occurs.

PROBLEM SOLUTIONS
12th USA MATHEMATICAL OLYMPIAD
MAY 3, 1983

1. On a given circle, six points A, B, C, D, E and F are chosen at random, independently and uniformly with respect to arc length. Determine the probability that the two triangles ABC and DEF are disjoint, i.e., having no common points.

Sol. Choose A. The two triangles will have no common point if and only if the remaining five points, reading clockwise from A, are arranged in the order $PQXYZ$, $PXYZQ$, or $XYZPQ$, where P,Q is a permutation of B,C, and X,Y,Z is a permutation of D,E,F. Since the five points B,C,D,E,F can be arranged in 5! equally likely ways, the probability we seek is

$$\frac{3\times 2!\times 3!}{5!} = \frac{3}{10} .$$

2. Prove that the roots of $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ cannot <u>all</u> be real if $2a^2 < 5b$.

Sol. Let the roots of the polynomial be denoted by a_i, $i=1,\ldots,5$. Then

$$a^2 = (-a)^2 = \left(\sum_{i=1}^{5} a_i\right)^2$$

$$= \sum_{i=1}^{5} a_i^2 + 2\sum_{i=1}^{5}\sum_{j=i+1}^{5} a_i a_j$$

$$= \sum_{i=1}^{5} a_i^2 + 2b.$$

Now suppose that all the roots are real. Then, by the arithmetic mean - geometric mean inequality

$$a_i a_j \le |a_i||a_j| \le \frac{a_i^2 + a_j^2}{2} .$$

It follows that

$$\sum_{i=1}^{5}\sum_{j=i+1}^{5} a_i a_j \le \sum_{i=1}^{5}\sum_{j=i+1}^{5}\left(\frac{a_i^2 + a_j^2}{2}\right)$$

$$= 2\sum_{i=1}^{5} a_i^2 ,$$

or equivalently, $\sum_{i=1}^{5} a_i^2 \ge \frac{b}{2}$.

Thus, from the first equation, if all the roots are real

$$a^2 = \sum_{i=1}^{5} a_i^2 + 2b \ge \frac{5}{2} b ,$$

and the proof is complete.

3. Each set of a finite family of subsets of a line is a union of two closed intervals. Moreover, any three of the sets of the family have a point in common. Prove that there is a point which is common to at least half of the sets of the given family.

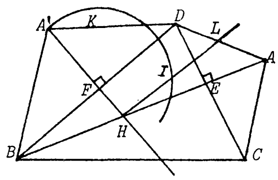
Sol. Let the members of the given family be denoted by A_k, $k=1,\ldots,n$. For each k, let $a_k = \min\{x \in A_k\}$, $b_k = \max\{x \in A_k\}$. Let s and t be such that $a_s = \max_k \{a_k\}$, and $b_t = \min_k \{b_k\}$. Because any two sets of the family have a point in common, $a_s \le b_t$.

We claim that either a_s belongs to at least half of the sets of the given family, or b_t belongs to at least half of the sets of the given family. For if not, there is a set in the family, say $A_r \equiv [a_r, c_r] \cup [d_r, b_r]$, which contains neither a_s nor b_t. Because $a_r \le a_s$ and a_s is not in A_r, $a_r \le c_r < a_s$. Similarly $b_t < d_r \le b_r$. It follows that $[a_r, c_r] \cap A_s = \emptyset$ and $[d_r, b_r] \cap A_t = \emptyset$, and therefore A_r, A_s, and A_t are sets of the family which have no point in common. This contradiction establishes the result.

4. Six segments S_1, S_2, S_3, S_4, S_5, and S_6 are given in a plane. These are congruent to the edges AB, AC, AD, BC, BD, and CD, respectively, of a tetrahedron $ABCD$. Show how to construct a segment congruent to the altitude of the tetrahedron from vertex A with straight-edge and compass.

Sol. First, construct a (partial) planar model of the tetrahedron as follows. Use segments S_4, S_5, and S_6

to construct ΔBCD. Then find points A' and A'' so that $A'B$ and $A'D$ are congruent to S_1 and S_3 respectively, and $A''C$ and $A''D$ are congruent to S_2 and S_3 respectively.



Second, fold the faces $A'BD$ and $A''CD$ into place by rotating them about BD and CD respectively. In this process, the locus of the point A' will project orthogonally onto the plane of the base, ΔBCD, to produce the line $A'F$ perpendicular to BD (see figure). Similarly, the projection of the locus of A'' will project to produce the line $A''E$ perpendicular to CD. The intersection of $A'F$ and $A''E$, say at H, is the foot of the altitude of the tetrahedron from the vertex A.

Finally to determine the altitude, construct a circle, K, with center F through A', and a half-line L through H perpendicular to $A'F$. Let I be the intersection of K and L. Then IH is congruent to the desired altitude.

5. Consider an open interval of length $1/n$ on the real number line where n is a positive integer. Prove that the number of irreducible fractions p/q, with $1 \le q \le n$, contained in the given interval is at most $(n+1)/2$.

Sol. Let S denote the set of all irreducible fractions p/q, satisfying $1 \le q \le n$. For each odd integer m, let S_m denote all those elements p/q in S whose denominator q has the form $2^k m$ for some integer k. Observe that each element of S is in exactly one S_m, m odd. Furthermore, because the least common denominator of any two elements of S_m is $\le n$, two distinct elements of S_m will differ by at least $1/n$. Thus, at most one element in each S_m can be in the given interval. This establishes the result since there are at most $(n+1)/2$ odd integers in the interval $[1,n]$.

PROBLEM SOLUTIONS
15th CANADIAN OLYMPIAD
1983

1. Find all positive integers w, x, y, z which satisfy
$$w! = x! + y! + z! \quad .$$

Sol. We may suppose that $x \le y \le z < w$. Then

$$w! = z!\left(\frac{x!}{z!} + \frac{y!}{z!} + 1\right) \le 3z! \quad (*)$$

so a necessary condition is that

$$\frac{w!}{z!} \le 3, \quad z < w \quad . \qquad (**)$$

The only solutions to (**) are $w = 3$, $z = 1$; $w = 2$, $z = 1$; $w = 3$, $z = 2$. Of these, only the last leads to a solution of (*), namely, $w = 3$, $x = y = z = 2$.

2. For each real number r, let T_r be the transformation of the plane that takes the point (x,y) into the point $(2^r x,\ r2^r x + 2^r y)$. Let F be the family of all such transformations, i.e., $F = \{T_r : r \text{ a real number}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.

Sol. I. Suppose $y = f(x)$ is a curve fixed by every T_r, and let $a = f(1)$. Consider $x > 0$, and let $r = \log_2 x$. Then
$$T_r(1,a) = (2^r,\ r2^r + 2^r a) = (x,\ x(\log_2 x + a)).$$
Now consider $x < 0$ and let $r = \log_2(-x)$ and $-b = f(-1)$. Then
$$T_r(-1,-b) = (-2^r,\ -r2^r - b2^r) = (x,\ x(\log_2(-x) + b)).$$
It follows that f must have the form
$$f(x) = \begin{cases} x(\log_2 x + a) & x > 0 \\ 0 & x = 0 \\ x(\log_2(-x) + b) & x < 0 \end{cases} .$$
Conversely, any such function, for arbitrary reals a and b, is invariant under the family F.

Sol. II. (based on calculus). Note

$$f'(x) = \lim_{r \to 0} \frac{f(2^r x) - f(x)}{2^r x - x}$$

$$= \lim_{r \to 0} \frac{r2^r x + 2^r f(x) - f(x)}{2^r x - x}$$

$$= \lim_{r \to 0} \left(\frac{r}{2^r - 1}\right) 2^r + \frac{f(x)}{x}$$

$$= \frac{1}{\ln 2} + \frac{f(x)}{x} .$$

Thus,

$$f'(x) - \frac{1}{x} f(x) = \frac{1}{\ln 2} .$$

Multiply each side by the integrating factor $1/x$ to get

$$\frac{d}{dx}\left(\frac{1}{x} f(x)\right) = \frac{1}{x \ln 2} .$$

Now integrate each side from 1 to x (for $x > 0$) to get

$$\frac{1}{x} f(x) - f(1) = \frac{\ln x}{\ln 2} , x > 0,$$

or equivalently,

$$f(x) = x(\log_2 x + f(1)) , x > 0.$$

A similar argument for $x < 0$ yields

$$f(x) = x(\log_2(-x) - f(-1)) , x < 0.$$

3. The area of a triangle is determined by the lengths of its sides. Is the volume of a tetrahedron determined by the areas of its faces?

Sol. No. Let S be an equilateral triangle and T be a right isosceles triangle, both of area 4. The lines joining pairwise the midpoints of the sides of each triangle divide it into four triangles of area 1. Folding along these lines, S turns into a regular tetrahedron with volume $V > 0$ while T turns into a degenerate tetrahedron (in fact a square) with zero volume. Yet both tetrahedra have faces all of area 1. One may reduce the right angle in T slightly so that the resulting tetrahedron is not degenerate. Its altitude and consequently its volume can be made arbitrarily close to zero.

4. Prove that for every prime number p, there are infinitely many positive integers n such that p divides $2^n - n$.

Sol. For $p = 2$, n can be any even (positive) integer. When $p > 2$, we know that $2^{p-1} \equiv 1 \bmod p$ (Fermat's Theorem). If $K(n) = (np - 1)(p-1)$, then $2^{K(n)} = (2^{p-1})^{np-1} \equiv 1 \bmod p$ and $K(n) \equiv 1 \bmod p$. Thus $2^{K(n)} - K(n)$ is divisible by p for each positive integer n.

5. The geometric mean (G.M.) of k positive numbers is defined to be the positive kth root of their product. For example, the G.M. of 3,4 and 18 is 6. Show that the G.M. of a set S of n positive numbers is equal to the G.M. of the G.M.'s of all nonempty subsets of S.

Sol. I. Let $X = \{x_1, x_2, \ldots, x_n\}$ and let T be a nonempty subset of S with k elements, $k > 0$. The G.M. of the elements of T has the form

$$x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}$$

where $t_1 + t_2 + \ldots + t_n = 1$ (the nonzero t_i are each equal to $1/k$). With this in mind, observe that the G.M. of the G.M.'s has the form

$$x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

where $m_1 + m_2 + \ldots + m_n = 1$ (each of the 2^n-1 nonempty subsets of S contribute $1/(2^n-1)$ to this sum). By symmetry, $m_1 = m_2 = \ldots = m_n = 1/n$ and the result follows.

Sol. II. A direct calculation of m_i can be carried out as follows. The number of k-subsets of S which contain x_i is $\binom{n-1}{k-1}$ and it follows that

$$m_i = \frac{1}{2^n-1} \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1}$$

$$= \frac{1}{2^n-1} \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} = \frac{1}{n} .$$

U.S. Postal Service

STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION

Required by 39 U.S.C. 3685)

1A. TITLE OF PUBLICATION: MATHEMATICS MAGAZINE

1B. PUBLICATION NO.: M A M C A 8

2. DATE OF FILING: Sept. 9, 1983

3. FREQUENCY OF ISSUE: Every two months except July and August

3A. NO. OF ISSUES PUBLISHED ANNUALLY: Five

3B. ANNUAL SUBSCRIPTION PRICE: Library $25.00; Indiv.Member $10.00

4. COMPLETE MAILING ADDRESS OF KNOWN OFFICE OF PUBLICATION *(Street, City, County, State and ZIP Code) (Not printers)*: 1529 Eighteenth St., N.W., Washington, D.C. 20036

5. COMPLETE MAILING ADDRESS OF THE HEADQUARTERS OF GENERAL BUSINESS OFFICES OF THE PUBLISHER *(Not printer)*: 1529 Eighteenth St., N.W., Washington, D.C. 20036

6. FULL NAMES AND COMPLETE MAILING ADDRESS OF PUBLISHER, EDITOR, AND MANAGING EDITOR *(This item MUST NOT be blank)*

PUBLISHER *(Name and Complete Mailing Address)*: The Mathematical Association of America, 1529 Eighteenth St.,N.W.,Washington,D.C. 20036

EDITOR *(Name and Complete Mailing Address)*: Doris Schattschneider, Moravian College, Bethlehem, Pennsylvania 18018

MANAGING EDITOR *(Name and Complete Mailing Address)*: Dr. Raoul Hailpern, SUNY at Buffalo, Buffalo, NY 14214

7. OWNER *(If owned by a corporation, its name and address must be stated and also immediately thereunder the names and addresses of stockholders owning or holding 1 percent or more of total amount of stock. If not owned by a corporation, the names and addresses of the individual owners must be given. If owned by a partnership or other unincorporated firm, its name and address, as well as that of each individual must be given. If the publication is published by a nonprofit organization, its name and address must be stated.) (Item must be completed.)*

FULL NAME	COMPLETE MAILING ADDRESS
The Mathematical Association of America, Inc.	1529 Eighteenth St., N.W. Washington, DC 20036

8. KNOWN BONDHOLDERS, MORTGAGEES, AND OTHER SECURITY HOLDERS OWNING OR HOLDING 1 PERCENT OR MORE OF TOTAL AMOUNT OF BONDS, MORTGAGES OR OTHER SECURITIES *(If there are none, so state)*

FULL NAME	COMPLETE MAILING ADDRESS
None	

9. FOR COMPLETION BY NONPROFIT ORGANIZATIONS AUTHORIZED TO MAIL AT SPECIAL RATES *(Section 423.12 DMM only)*
The purpose, function, and nonprofit status of this organization and the exempt status for Federal income tax purposes *(Check one)*

(1) [X] HAS NOT CHANGED DURING PRECEDING 12 MONTHS

(2) [ ] HAS CHANGED DURING PRECEDING 12 MONTHS

(If changed, publisher must submit explanation of change with this statement.)

10. EXTENT AND NATURE OF CIRCULATION	AVERAGE NO. COPIES EACH ISSUE DURING PRECEDING 12 MONTHS	ACTUAL NO. COPIES OF SINGLE ISSUE PUBLISHED NEAREST TO FILING DATE
A. TOTAL NO. COPIES *(Net Press Run)*	10,890	10,573
B. PAID CIRCULATION 1. Sales through dealers and carriers, street vendors and counter sales	5	5
2. Mail Subscription	8,595	8,696
C. TOTAL PAID CIRCULATION *(Sum of 10B1 and 10B2)*	8,600	8,701
D. FREE DISTRIBUTION BY MAIL, CARRIER OR OTHER MEANS SAMPLES, COMPLIMENTARY, AND OTHER FREE COPIES	61	61
E. TOTAL DISTRIBUTION *(Sum of C and D)*	8,661	8,762
F. COPIES NOT DISTRIBUTED 1. Office use, left over, unaccounted, spoiled after printing	2,229	1,811
2. Return from News Agents	none	none
G. TOTAL *(Sum of E, F1 and 2—should equal net press run shown in A)*	10,890	10,573

11. I certify that the statements made by me above are correct and complete

SIGNATURE AND TITLE OF EDITOR, PUBLISHER, BUSINESS MANAGER, OR OWNER: Alfred B. Willcox, Executive Director

PS Form 3526, July 1982 *(See instruction on reverse)*

THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, N.W.
Washington, DC 20036

MATHEMATICS MAGAZINE VOL. 57, NO. 1, JANUARY 1984